Tier 1 Analysis Exam

January 2000

1. Let Ω be an open set in $\mathbb{R}^2$. Let u be a real-valued function on Ω. Suppose that for each point $a \in \Omega$ the partial derivatives $u_x(a)$ and $u_y(a)$ exist and are equal to zero.
   (i) Prove that u is locally constant, *i.e.* for every point in Ω there is a neighborhood on which u is a constant function.
   (ii) Prove that if Ω is connected, then u is a constant function on Ω .

2. Let S be the surface in the Euclidean space $\mathbb{R}^3$ given by the equation $x^2 + y^2 - z^2 = 1$, $0 \le z \le 1$, oriented so that the normal vector points away from the z-axis. Find $\int\limits_S \mathbf{F} \cdot \mathbf{dS}$, where $\mathbf{F}$ is the vector field defined by
$$\mathbf{F}(x, y, z) = (-xy^2 + z^5,\ -x^2 y,\ (x^2 + y^2)z) \quad .$$

3. Let $f(x) = e^x - \cos x$ for $x \in \mathbb{R}$.
   (i) Show that on a neighborhood around $x = 0$, f has an inverse function g with $g(0) = 0$.
   (ii) Compute $g''(0)$.
   (iii) Show that there exists $a > 0$ such that $f : (-a, \infty) \to (f(-a), \infty)$ is a homeomorphism.

4. For positive numbers $k_1,\ k_2,\ k_3,\ \ldots$ we define $[k_1] = \frac{1}{k_1}$, $[k_1, k_2] = \frac{1}{k_1 + [k_2]}$, $[k_1, k_2, k_3] = \frac{1}{k_1 + [k_2, k_3]}$, and inductively, $[k_1, \ldots, k_{n+1}] = \frac{1}{k_1 + [k_2, \ldots, k_{n+1}]}$. Prove that $\lim\limits_{n\to\infty} [k_1, \ldots, k_n]$ exists if $k_n \ge 2$ for all n.

5. Two circular holes of radius 1 in are drilled from the centers of two faces of a solid cube of volume 64 in^3 . Compute the volume of the remaining solid.

6. Let $\varphi_1,\ \varphi_2,\ \varphi_3,\ \ldots$ be non-negative continuous functions on $[-1, 1]$ such that
   (i) $\int_{-1}^{1} \varphi_k(t)dt = 1$ for $k = 1, 2, 3, \ \ldots$;
   (ii) for every $\delta \in (0, 1)$ $\lim\limits_{k\to\infty} \varphi_k = 0$ uniformly on $[-1, -\delta] \cup [\delta, 1]$.

Prove that for every continuous function $f : [-1,1] \to \mathbb{R}$ we have

$$\lim_{k\to\infty} \int_{-1}^{1} f(t)\varphi_k(t)dt = f(0) \quad .$$

7. Suppose $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, and let

$$c_n = \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} \quad .$$

Prove that $\lim\limits_{n\to\infty} c_n = ab$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$. Prove that there exist positive constants A and B such that

$$|f(x)| \le A|x| + B \quad \text{for all } x \in \mathbb{R} \quad .$$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 1$. Prove that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} x_n = \infty$ and $\lim\limits_{n\to\infty} f'(x_n) = 1$.