1. Let $\Omega$ be an open set in $\mathbb{R}^2$. Let $u$ be a real-valued function on $\Omega$. Suppose that for each point $a \in \Omega$ the partial derivatives $u_x(a)$ and $u_y(a)$ exist and are equal to zero.

(i) Prove that $u$ is locally constant, i.e. for every point in $\Omega$ there is a neighborhood on which $u$ is a constant function.

(ii) Prove that if $\Omega$ is connected, then $u$ is a constant function on $\Omega$.

2. Let $S$ be the surface in the Euclidean space $\mathbb{R}^3$ given by the equation $x^2 + y^2 - z^2 = 1$, $0 \leq z \leq 1$, oriented so that the normal vector points away from the $z$-axis. Find $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ is the vector field defined by

$$\mathbf{F}(x, y, z) = (-xy^2 + z^5, -x^2y, (x^2 + y^2)z).$$

3. Let $f(x) = e^x - \cos x$ for $x \in \mathbb{R}$.

(i) Show that on a neighborhood around $x = 0$, $f$ has an inverse function $g$ with $g(0) = 0$.

(ii) Compute $g''(0)$.

(iii) Show that there exists $a > 0$ such that $f : (-a, \infty) \to (f(-a), \infty)$ is a homeomorphism.

4. For positive numbers $k_1$, $k_2$, $k_3$, ... we define $[k_1] = \frac{1}{k_1}$, $[k_1, k_2] = \frac{1}{k_1 + [k_2]}$, $[k_1, k_2, k_3] = \frac{1}{k_1 + [k_2, k_3]}$, and inductively, $[k_1, \ldots, k_{n+1}] = \frac{1}{k_1 + [k_2, \ldots, k_{n+1}]}$. Prove that $\lim_{n \to \infty} [k_1, \ldots, k_n]$ exists if $k_n \geq 2$ for all $n$.

5. Two circular holes of radius 1 in are drilled from the centers of two faces of a solid cube of volume 64 in$^3$. Compute the volume of the remaining solid.

6. Let $\varphi_1$, $\varphi_2$, $\varphi_3$, ... be non-negative continuous functions on $[-1, 1]$ such that

(i) $\int_{-1}^1 \varphi_k(t)dt = 1$ for $k = 1, 2, 3, \ldots$;

(ii) for every $\delta \in (0, 1)$ $\lim_{k \to \infty} \varphi_k = 0$ uniformly on $[-1, -\delta] \cup [\delta, 1]$.  

Prove that for every continuous function \( f : [-1, 1] \to \mathbb{R} \) we have
\[
\lim_{k \to \infty} \int_{-1}^{1} f(t) \varphi_k(t) \, dt = f(0).
\]

7. Suppose \( \lim_{n \to \infty} a_n = a \), \( \lim_{n \to \infty} b_n = b \), and let
\[
c_n = \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}.
\]

Prove that \( \lim_{n \to \infty} c_n = ab \).

8. Let \( f : \mathbb{R} \to \mathbb{R} \) be a uniformly continuous function on \( \mathbb{R} \). Prove that there exist positive constants \( A \) and \( B \) such that
\[
|f(x)| \leq A|x| + B \quad \text{for all } x \in \mathbb{R}.
\]

9. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Suppose \( \lim_{x \to \infty} \frac{f(x)}{x} = 1 \). Prove that there exists a sequence \( \{x_n\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} x_n = \infty \) and \( \lim_{n \to \infty} f'(x_n) = 1 \).