Tier 1 Analysis Examination

August 1998

1. Consider the sequence of functions $f_k(x) := \{\sin(kx)\}$, $k = 1, 2, \ldots$, and observe that $\sin(kx) = 0$ if $x = m\pi/k$ for all integers m. Given an arbitrary interval $[a, b]$, show that $\{f_k\}$ has no subsequence that converges uniformly on $[a, b]$.

2.
   (a) Given a sequence of functions f_k defined on $[0, 1]$, define what it means for $\{f_k\}$ to be equicontinuous.

   (b) Let $G(x, y)$ be a continuous function on $\mathbf{R}^2$ and suppose for each positive integer k, that g_k is a continuous function defined on $[0, 1]$ with the property that $|g_k(y)| \leq 1$ for all $y \in [0, 1]$. Now define

$$f_k(x) := \int_0^1 g_k(y) G(x, y)\, dy.$$

   Prove that the sequence $\{f_k\}$ is equicontinuous on $[0, 1]$.

3. Let $\Omega \subset \mathbf{R}^n$ be an open connected set and let $\Omega \xrightarrow{f} \Omega$ be a C^1 transformation with the property that determinant of its Jacobian matrix, $|Jf|$, never vanishes. That is, $|Jf(x)| \neq 0$ for each $x \in \Omega$. Assume also that $f^{-1}(K)$ is compact whenever $K \subset \Omega$ is a compact set. Prove that $f(\Omega) = \Omega$.

4. Let $G(x, y)$ be a continuous function defined on $\mathbf{R}^2$. Consider the function f defined for each $t > 0$ by

$$f(t) := \int\int_{x^2+y^2<t^2} \frac{G(x, y)}{\sqrt{t^2 - x^2 - y^2}} dx\, dy.$$

   Prove that

$$\lim_{t \to 0^+} f(t) = 0.$$

5. Let $(X, \boldsymbol{d})$ be a compact metric space and let $\mathcal{G}$ be an arbitrary family of open sets in X. Prove that there is a number $\lambda > 0$ with the property that if $x, y \in X$ are points with $\boldsymbol{d}(x, y) < \lambda$, then there exists an open set $U \in \mathcal{G}$ such that both x and y belong to U.

6. Let $\Gamma := \{(x, y, z) \in \mathbf{R}^3 : e^{xy} = x, x^2 + y^2 + z^2 = 10\}$. The Implicit Function theorem ensures that Γ is a curve in some neighborhood of the point $p = (e, \frac{1}{e}, \sqrt{10 - e^2 - \frac{1}{e^2}})$. That is, there is open interval $I \subset \mathbf{R}^1$ and a C^1 mapping $I \xrightarrow{\gamma} \Gamma$ such that $\gamma(0) = p$. Find a unit vector v such that $v = \pm\frac{\gamma'(0)}{|\gamma'(0)|}$.

7. Suppose that a hill is described as $\{(x, y, z) \in \mathbf{R}^3 : (x, y, f(x, y))\}$ where $f(x, y) = x^3 + x - 4xy - 2y^2$. Suppose that a climber is located at $p = (1, 2, -14)$ on the hill and wants to move from p to another location on the hill without changing elevation. In which direction should the climber proceed from p? Express your answer in terms of a vector and completely justify your answer.

8. Suppose g and f_k $(k = 1, 2, \dots)$ are defined on $(0, \infty)$, are Riemann integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_k| \le g$, $f_k \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)\, dx < \infty.$$

Prove that

$$\lim_{k \to \infty} \int_0^\infty f_k(x)\, dx = \int_0^\infty f(x)\, dx.$$