ANALYSIS EXAMINATION

January, 1998

Instructions: Answer all seven questions. Each of the seven questions is equally weighted.
Notation: $\mathbf{R}$ denotes the set of all real numbers.

1. State whether each of the following limit exists, and prove your assertions.

   (a) $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4+y^4}$ (b) $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4+y^6}$

2. Suppose $f:\mathbf{R}\to\mathbf{R}$ is a uniformly continuous function on $\mathbf{R}$. If

$$f_k(x) = k\int_x^{x+(1/k)} f(t)\,dt \qquad \text{for } x\in\mathbf{R} \text{ and } k=1,\ 2,\ 3,\ \ldots,$$

   prove that the sequence $\{f_k\}$ converges to f uniformly on $\mathbf{R}$.

3. Compute the surface integral $\iint_S (x^2+y^2)\,dA$, where S is the boundary of the set $\{(x,y,z)\in\mathbf{R}^3 : \sqrt{x^2+y^2}\le z\le 1\}$, and dA denotes the surface area element.

4. Let $\pi_1:\mathbf{R}^2\to\mathbf{R}$ and $\pi_2:\mathbf{R}^2\to\mathbf{R}$ be the projection maps

$$\pi_1(x,y)=x, \quad \pi_2(x,y)=y, \qquad \text{for } (x,y)\in\mathbf{R}^2,$$

   and let S be the horizontal strip $S=\{(x,y)\in\mathbf{R}^2 : -1\le y\le 1\}$. State whether each of the following assertions is TRUE or FALSE, and prove your assertions.

   (a) If E is a closed subset of $\mathbf{R}^2$ such that $E\subset S$, then the image $\pi_1(E)$ must be a closed subset of $\mathbf{R}$.

   (b) If E is a closed subset of $\mathbf{R}^2$ such that $E\subset S$, then the image $\pi_2(E)$ must be a closed subset of $\mathbf{R}$.

5. If f is a continuous function on $[0,1]$, prove that

$$\lim_{t\uparrow 1}\left[(1-t)\sum_{k=0}^{\infty} t^k f(t^k)\right] = \int_0^1 f(x)\,dx.$$

6. Let Ω be a bounded, connected open set in $\mathbf{R}^n$, and let f be a continuous real-valued function on the closure of Ω. Suppose that f is of class C^∞ on the set Ω, and suppose that for each point $p\in\Omega$ there is at least one index $i\in\{1,2,\ldots,n\}$ such that

$$\frac{\partial^2 f}{\partial x_i^2}(p) < 0.$$

   If

$$f(p)\ge 0 \qquad \text{for every point } p \text{ in the boundary of } \Omega,$$

   prove that

$$f(p)\ge 0 \qquad \text{for every point } p \text{ in } \Omega.$$

7. Let Ω be a convex open set in $\mathbf{R}^2$. Let $f:\Omega\to\mathbf{R}$ and $g:\Omega\to\mathbf{R}$ be functions of class C^∞, and assume that for each point $p\in\Omega$ we have

$$\frac{\partial f}{\partial x}(p)\ge 5, \qquad \frac{\partial g}{\partial y}(p)\ge 5, \qquad \left|\frac{\partial f}{\partial y}(p)\right|\le 1, \qquad \left|\frac{\partial g}{\partial x}(p)\right|\le 1.$$

   Define $T:\Omega\to\mathbf{R}^2$ by $T(p)=(f(p),g(p))$ for each point $p\in\Omega$. Prove that the image $T(\Omega)$ is an open subset of $\mathbf{R}^2$, and that T is a one-to-one mapping from Ω onto $T(\Omega)$.