You should attempt all nine of the following problems. Good luck!

1. Let $X$ be the metric space 
   
   $X = \{(x, y) \in \mathbb{R}^2 : y \geq |x|^{2/3}\}$
   
   with the usual Euclidean distance, and define $f : X \rightarrow \mathbb{R}$ by $f(x, y) = \frac{xy^3}{x^4 + y^4}$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Decide whether or not $f$ is continuous at $(0, 0)$, and prove your answer by using the $\varepsilon - \delta$ definition of continuity. Is $f$ continuous at $(0, 0)$ when considered as a mapping from $\mathbb{R}^2$ into $\mathbb{R}$? Prove your answer.

2. Define $g : [-1, 1] \rightarrow \mathbb{R}$ by $g(x) = (-1)^k / k^2$ for $|x| \in (1/(k + 1), 1/k]$, $k = 1, 2, \ldots$, and $g(0) = 0$. Decide whether or not $g$ is differentiable at $0$, and prove your answer.

3. Let $\{a_n\}_{n=0}^{\infty}$ be the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$. (Thus $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$.) Show that the series $\sum_{n=0}^{\infty} \frac{1}{a_n}$ converges.

4. Compute $\int_{\Phi} \text{curl} F \cdot N \, dA$, where $F$ is the vector field $F(x, y, z) = \frac{(-z, y, x)}{\sqrt{x^2 + y^2 + 1}}$, $\Phi : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ is the surface $\Phi(r, \theta) = (r \cos \theta, r^2, r \sin \theta)$, $N$ is a unit normal vector on $\Phi$, and $dA$ is the surface area element.

5. Let $E$ be an open set in $\mathbb{R}^n$, and let $F : E \rightarrow \mathbb{R}^n$ be $C^1$. Show that, if the function $|F|^2$ has a nonzero relative minimum at a point $x_0 \in E$, then the linear transformation $F'(x_0)$ must be singular.

6. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous, and assume that $\lim_{x \to -\infty} f(x)$ exists and is a finite number $L$. What can be said about $\lim_{n \to -\infty} \int_{n}^{1} f(nx) \, dx$?

Prove your answer.

7. Let $A$ be the set of real numbers in $[0, 1]$ whose decimal expansions contain only the digits 3 and 8. Is $A$ countable? Is $A$ dense in $[0, 1]$? Is $A$ closed? Prove your answers.

8. Let $E \subset \mathbb{R}^2$ be open and nonempty. Prove that there is no one-to-one, $C^1$ function mapping $E$ into $\mathbb{R}$.

9. Let $E \subset \mathbb{R}^2$ be open, and let $F : E \rightarrow \mathbb{R}$ have continuous second order partial derivatives in $E$. Denote by $f''$ the matrix of second partial derivatives $[f_{xx} \ f_{xy} \ f_{yx} \ f_{yy}]$.
   
   a. Show that the set of points in $E$ at which $f''$ has repeated eigenvalues is closed relative to $E$.
   
   b. Suppose that $f''$ is positive definite in $E$; that is, suppose that, for each $x \in E$ and $h \in \mathbb{R}^2 - \{0\}$, $(f''(x)h) \cdot h > 0$. Show that, for any compact subset $K \subset E$, there is a positive constant $\varepsilon$ such that
   
   $(f''(x)h) \cdot h \geq \varepsilon |h|^2$
   
   for all $x \in K$ and all $h \in \mathbb{R}^2$. 