Department of Mathematics–Indiana University

Analysis Qualifying Exam

August, 1996

You should attempt all nine of the following problems. Good luck!

1. Let X be the metric space
$$X = \{(x, y) \in \mathbf{R}^2 : y \geq |x|^{2/3}\}$$
with the usual Euclidean distance, and define $f : X \to \mathbf{R}$ by $f(x,y) = \dfrac{xy^3}{x^4 + y^4}$ for $(x, y) \neq (0,0)$, and $f(0,0) = 0$. Decide whether or not f is continuous at $(0,0)$, and prove your answer by applying the $\epsilon - \delta$ definition of continuity. Is f continuous at $(0,0)$ when considered as a mapping from $\mathbf{R}^2$ into $\mathbf{R}$? Prove your answer.

2. Define $g : [-1,1] \to \mathbf{R}$ by $g(x) = (-1)^k/k^2$ for $|x| \in (1/(k+1), 1/k]$, $k = 1, 2, \ldots$, and $g(0) = 0$. Decide whether or not g is differentiable at 0, and prove your answer.

3. Let $\{a_n\}_{n=0}^{\infty}$ be the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$. (Thus $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$.) Show that the series $\sum_{n=0}^{\infty} \dfrac{1}{a_n}$ converges.

4. Compute $\int_{\Phi} \operatorname{curl} F \cdot N dA$, where F is the vector field $F(x, y, z) = \dfrac{(-z, y, x)}{\sqrt{x^2 + z^2 + 1}}$, $\Phi : [0, 1] \times [0, 2\pi] \to \mathbf{R}^3$ is the surface $\Phi(r, \theta) = (r\cos\theta, r^2, r\sin\theta)$, N is a unit normal vector on Φ, and dA is the surface area element.

5. Let E be an open set in $\mathbf{R}^n$, and let $F : E \to \mathbf{R}^n$ be C^1. Show that, if the function $|F|^2$ has a nonzero relative minimum at a point $x_0 \in E$, then the linear transformation $F'(x_0)$ must be singular.

6. Let $f : [0, \infty) \to \mathbf{R}$ be continuous, and assume that $\lim_{x\to\infty} f(x)$ exists and is a finite number L. What can be said about
$$\lim_{n\to\infty} \int_0^1 f(nx)\, dx \ ?$$
Prove your answer.

7. Let A be the set of real numbers in $[0, 1]$ whose decimal expansions contain only the digits 3 and 8. Is A countable? Is A dense in $[0,1]$? Is A closed? Prove your answers.

8. Let $E \subset \mathbf{R}^2$ be open and nonempty. Prove that there is no one-to-one, C^1 function mapping E into $\mathbf{R}$.

9. Let $E \subset \mathbf{R}^2$ be open, and let $F : E \to \mathbf{R}$ have continuous second order derivatives in E. Denote by f'' the matrix of second partial derivatives $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.
   a. Show that the set of points in E at which f'' has repeated eigenvalues is closed relative to E.
   b. Suppose that f'' is positive definite in E; that is, suppose that, for each $x \in E$ and $h \in \mathbf{R}^2 - \{0\}$, $(f''(x)h) \cdot h > 0$. Show that, for any compact subset $K \subset E$, there is a positive constant ϵ such that
$$(f''(x)h) \cdot h \geq \epsilon |h|^2$$
   for all $x \in K$ and all $h \in \mathbf{R}^2$.