Tier 1 Algebra Exam

August 2024

You may answer as many questions as you like and each question is worth 10 points total. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. Write the problem number on each page. At the end, assemble your solutions with the problems in increasing order. There are eight questions. You have 4 hours.

The set of $n \times n$-matrices with entries in a field F is denoted by $M_n(F)$.

1. Set

$$A = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(a) (7 pts) Find the Jordan canonical form of A as a matrix in $M_4(\mathbb{C})$.

(b) (3 pts) Let $R = \{a_0 + a_1 A + a_2 A^2 + \ldots + a_n A^n \mid n \geq 0, a_i \in \mathbb{R}\}$, which is a subset of the ring $M_4(\mathbb{R})$ of real 4×4-matrices. Is R a field?

2. (10 pts) Let A be the following matrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Find an orthogonal matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

3. Let V be an n-dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation which has the property that $T \circ T = T$.

(a) (5 pts) Show that the null space $\ker(T)$ and the image $\operatorname{im}(T)$ are complementary subspaces of V, that is, $V = \ker(T) + \operatorname{im}(T)$ and $\ker(T) \cap \operatorname{im}(T) = \{0\}$.

(b) (5 pts) Show that there is an ordered basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V such that the matrix of T with respect to $\mathcal{B}$ has the form

$$\left(\begin{array}{c|c} \mathbf{I}_r & \mathbf{O}_{r,s} \\ \hline \mathbf{O}_{s,r} & \mathbf{O}_{s,s} \end{array}\right)$$

where $s = n - r$, $\mathbf{I}_r$ is the $r \times r$-identity matrix, and $\mathbf{O}_{k,\ell}$ is the $k \times \ell$-zero matrix.

4. (10 pts) Show that if the automorphism group of a group G is cyclic, then G is abelian.

5. (a) (3 pts) Find the size of the group of inner automorphisms $\mathrm{Inn}(\mathcal{S}_3)$ of $\mathcal{S}_3$.

   (b) (7 pts) Find the size of the automorphism group $\mathrm{Aut}(\mathcal{S}_3)$ of $\mathcal{S}_3$.

6. (a) (5 pts) Let G be a group. Show that if H is a subgroup of G of index n there is a homomorphism $\phi_H : G \to \mathcal{S}_n$ such that $\{x \in G \mid \phi_H(x)(1) = 1\} = H$. ($\mathcal{S}_n$ denotes the symmetric group on the numbers $1, \ldots, n$.)

   (b) (5 pts) Now suppose that G is a finitely generated group. Show the set of all subgroups of G of index n is a finite set.

7. (10 pts) Find the monic polynomial over $\mathbb{Q}$ of smallest degree with $\sqrt{2} + \sqrt{3}$ as a root.

8. (10 pts) Let $P < \mathbb{Q}[x, y, z]$ be a prime ideal, and let R be the quotient ring $\mathbb{Q}[x, y, z]/P$. The ring homomorphism $\mathbb{Q} \to \mathbb{Q}[x, y, z] \to R$ also endows R with the structure of a vector space over $\mathbb{Q}$. Show that if R is finite dimensional as a vector space over $\mathbb{Q}$ then P must be a maximal ideal.