## Tier 1 Algebra - January 2020

All problems carry equal weight. All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page.

1. Let

$$
A=\left[\begin{array}{lll}
1 & 1 & 1 \\
0 & 2 & 1 \\
0 & 0 & 3
\end{array}\right]
$$

(i) Find the eigenvectors of $A$.
(ii) Find $A^{100}$.

Remark: When writing down the entries of $A^{100}$ use exponential notation, e.g., $2^{100}+3^{100}$, or similar.
2. Let $A$ be an $n \times n$-matrix. Show that

$$
\operatorname{det}(A+I)=\sum_{B} \operatorname{det} B
$$

where the sum is over all the principal minors of $A$.
[The principal minors are the matrices $B$ obtained from $A$ by removing rows $1 \leq$ $i_{1}<\ldots<i_{r} \leq n$ and columns $i_{1}<\ldots<i_{r}$ indexed by the same numbers, where $r$ runs through $\{0, \ldots, n\}$, and the determinant of the "empty matrix" is 1.]
3. Let $\phi: \mathbb{C}^{7} \rightarrow \mathbb{C}^{7}$ be a be a $\mathbb{C}$-linear endomorphism with $\operatorname{dim}\left(\operatorname{ker}\left(\phi^{3}\right)\right)=5$ and $\phi^{7}=0$. What are the possible Jordan canonical forms of $\phi$ ?
4. Let $V$ be a finite-dimensional vector space over a field $k$, and let $U \subsetneq V$ be a proper subspace. Suppose $\phi: V \rightarrow V$ is an endomorphism of $V$ which has the following properties:

- $\left.\phi\right|_{U}=0$, i.e., $U \subset \operatorname{ker}(\phi)$;
- the induced map $\bar{\phi}: V / U \rightarrow V / U$, defined by $\bar{\phi}(v+U):=\phi(v)+U$, is the zero map.
(i) Show that $\phi^{2}=\phi \circ \phi$ is the zero map.
(ii) Is $\phi$ necessarily zero too? If so, give a proof, if not, give a counterexample.

5. Let $G$ be a subgroup of $S_{6}$, the symmetric group on 6 elements. Suppose $G$ contains a 6-cycle. Prove that $G$ has a normal subgroup $H$ of index 2 .
[Hint: You may use the sign homomorphism.]
6. (i) Let $G$ be a finite group. Show that the size (i.e., cardinality) of every conjugacy class in $G$ divides the order of $G$.
(ii) Use (i) to determine all finite groups $G$ such that $G$ has exactly two conjugacy classes.
7. Determine if the following assertions are true or false. Justify why or give a counterexample.
(i) Every subgroup of index 3 in a group is normal.
(ii) Every group is isomorphic to a subgroup of a symmetric group.
(iii) If a homomorphism $\phi: G \rightarrow H$ is onto then there is a subgroup $K$ of $G$ with $H \cong K$.
(iv) $\mathbb{Z} / 6 \mathbb{Z} \times \mathbb{Z} / 6 \mathbb{Z} \cong \mathbb{Z} / 4 \mathbb{Z} \times \mathbb{Z} / 9 \mathbb{Z}$.
8. Show that the group $R^{*}$ of invertible elements in the ring $R=\mathbb{Z} / 105 \mathbb{Z}$, is isomorphic to $\mathbb{Z} / 12 \mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z}$.
[Recall that an element $x$ of a commutative ring $R$ is called invertible, if there is an element $y$ such that $x y=1_{R}$.]
9. Let $\omega=e^{2 \pi i / 5}$ be a $5^{\text {th }}$ root of unity. The field $L=\mathbb{Q}(\omega)$ contains $K=\mathbb{Q}(\sqrt{5})$ because $\omega+\omega^{-1}=\frac{-1+\sqrt{5}}{2}$.
(i) Find the minimal polynomial for $\omega$ over $\mathbb{Q}$. (Argue from first principles. Do not use "well-known" facts about cyclotomic polynomials.)
(ii) Find the minimal polynomial for $\omega$ over $\mathbb{Q}(\sqrt{5})$.
10. Let $P$ be a prime ideal in a commutative ring $R$ with 1 . Let

$$
f(x)=x^{n}+a_{1} x^{n-1}+\ldots+a_{n-1} x+a_{n} \in R[x]
$$

be a non-constant monic polynomial with coefficients in $R$. Suppose that all coefficients $a_{1}, \ldots, a_{n}$ are in $P$, and that $f(x)=g(x) h(x)$, for some non-constant monic polynomials $g(x), h(x) \in R[x]$. Then show that the constant term of $f(x)$ is in $P^{2}$, the ideal generated by all elements of the form $a b$, where $a, b \in P$.

