Algebra Tier 1

August 2019

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

1. Suppose H_1 and H_2 are subgroups of a finite group G. Prove that

$$[G : H_1 \cap H_2] \leq [G : H_1][G : H_2],$$

with equality if and only if every element of G can be written $h_1 h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$. Do **not** assume that G is abelian.

2. Let G be a group and $G^2 < G$ the subgroup generated by all elements in G of the form g^2. Show that G^2 is normal in G and G/G^2 is an abelian group in which every element other than the identity has order 2.

3. Let p and q be two distinct prime numbers. Find the minimal polynomial of $\sqrt{p} + \sqrt{q}$ over $\mathbb{Q}$, and prove that it is indeed the minimal polynomial of $\sqrt{p} + \sqrt{q}$.

4. a. Show that $\mathbb{Z}[x]/(x^2 + x + 1, 5)$ is a field. How many elements does it have?

   b. Show that $\mathbb{Z}[x, y]/(xy - 1)$ and $\mathbb{Z}[t]$ are not isomorphic rings.

5. Prove that there is an isomorphism of rings

$$\mathbb{C}[x, y]/(x - x^3 y) \to \mathbb{C}[y] \oplus \mathbb{C}[u, 1/u] \ ,$$

where $\mathbb{C}[u, 1/u]$ is the ring of Laurent polynomials $\sum_{i=-m}^{n} a_i u^i$ $(m, n \geq 0)$ with complex coefficients.

6. Let F be a field of characteristic $p > 0$, n a positive integer, and N a nilpotent $n \times n$ matrix with entries in F (this means $N^k = 0$ for some positive integer k). Prove that $I + N$ is invertible, that it is of finite order in $\mathrm{GL}_n(F)$, and that the order is a power of p.

7. Let V be a finite-dimensional vector space and $T \colon V \to V$ a linear transformation. Prove that

$$\dim \ker T^2 \geq \frac{\dim \ker T + \dim \ker T^3}{2}.$$

8. Let $\mathbb{F}_q$ be a field with q elements. Let

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}_q^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}.$$

How many vector subspaces of dimension 2 of V contain the vector $(1, 1, -1, -1)$?

9. Give an explicit formula for

$$\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}^n$$

in terms of the positive integer n.