All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

1. Suppose \( H_1 \) and \( H_2 \) are subgroups of a finite group \( G \). Prove that

\[
[G : H_1 \cap H_2] \leq [G : H_1][G : H_2],
\]

with equality if and only if every element of \( G \) can be written \( h_1h_2 \) for some \( h_1 \in H_1 \) and \( h_2 \in H_2 \). Do not assume that \( G \) is abelian.

2. Let \( G \) be a group and \( G^2 < G \) the subgroup generated by all elements in \( G \) of the form \( g^2 \). Show that \( G^2 \) is normal in \( G \) and \( G/G^2 \) is an abelian group in which every element other than the identity has order 2.

3. Let \( p \) and \( q \) be two distinct prime numbers. Find the minimal polynomial of \( \sqrt{p} + \sqrt{q} \) over \( \mathbb{Q} \), and prove that it is indeed the minimal polynomial of \( \sqrt{p} + \sqrt{q} \).

4. a. Show that \( \mathbb{Z}[x]/(x^2 + x + 1, 5) \) is a field. How many elements does it have?

   b. Show that \( \mathbb{C}[x, y]/(xy - 1) \) and \( \mathbb{C}[t] \) are not isomorphic rings.

5. Prove that there is an isomorphism of rings

\[
\mathbb{C}[x, y]/(x - x^3y) \to \mathbb{C}[y] \oplus \mathbb{C}[u, 1/u],
\]

where \( \mathbb{C}[u, 1/u] \) is the ring of Laurent polynomials \( \sum_{i=-m}^{n} a_i u^i \) \( (m, n \geq 0) \) with complex coefficients.

6. Let \( F \) be a field of characteristic \( p > 0 \), \( n \) a positive integer, and \( N \) a nilpotent \( n \times n \) matrix with entries in \( F \) (this means \( N^k = 0 \) for some positive integer \( k \)). Prove that \( I + N \) is invertible, that it is of finite order in \( \text{GL}_n(F) \), and that the order is a power of \( p \).

7. Let \( V \) be a finite-dimensional vector space and \( T : V \to V \) a linear transformation. Prove that

\[
\dim \ker T^2 \geq \frac{\dim \ker T + \dim \ker T^3}{2}.
\]

8. Let \( \mathbb{F}_q \) be a field with \( q \) elements. Let

\[
V = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}_q^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}.
\]

How many vector subspaces of dimension 2 of \( V \) contain the vector \( (1, 1, -1, -1) \)?

9. Give an explicit formula for

\[
\begin{pmatrix}
3 & 1 \\
-2 & 0
\end{pmatrix}^n
\]

in terms of the positive integer \( n \).