Tier 1 Algebra Exam

August 2020

All your answers should be justified (except where specifically indicated otherwise): A correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. Each problem is worth 10 points.

1. Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces. Show that there are bases $\mathcal{B}$ of V and $\mathcal{C}$ of W so that the matrix M of T relative to these bases has $M_{ii} = 1$ for $1 \leq i \leq \dim T(V)$ and all other entries zero.

2. Let H be the subgroup of $\mathbb{Z}^2$ generated by $\{(5, 15), (10, 5)\}$. Find an explicit isomorphism between $\mathbb{Z}^2/H$ and the product of cyclic groups.

3. Let $\alpha, \beta \in \mathbb{C}$ be complex numbers with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3 = [\mathbb{Q}(\beta) : \mathbb{Q}]$. Determine the possibilities for $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$. Give an example, without proof, of each case.

4. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

   (a) Find a 2×2 matrix P such that PAP^{-1} is upper triangular.

   (b) Find a formula for A^n.

5. Let H be a proper subgroup of a finite group G. Show that G cannot be the union of the conjugates of H, i.e., $G \neq \cup_{g \in G}\, gHg^{-1}$.

6. Let $R \subset S$ be commutative rings with 1. An element $u \in S$ is said to be integral over R if there is a monic polynomial $f(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_0 \in R[x]$ such that $f(u) = 0$. Show that if S is a field and every element of S is integral over R, then R is a field.

7.

   (a) Find a 2×2 real matrix so that $A^5 = I$ but $A \neq I$.

   (b) Show that there is no 2×2 integral matrix so that $A^5 = I$ but $A \neq I$.

(c) Find a 4×4 integral matrix so that $A^5 = I$ but $A \neq I$.

8. Identify all isomorphism classes of groups of order 20 having a unique subgroup of order 5 and an element of order 4.

9. Let R be a commutative ring with 1. Let $U \subset R$ be a subset containing 1 such that $u \cdot v \in U$ for all $u, v \in U$. Let J be an ideal of R such that

   (a) $J \cap U = \emptyset$, and

   (b) if I is an ideal of R strictly containing J, then $I \cap U \neq \emptyset$.

   Show that J is a prime ideal.

10. Prove or give a counterexample.

   (a) Every element of a finite field is the sum of two squares.

   (b) Every element of a finite field is the sum of two cubes.