TIER 1 ALGEBRA, JANUARY 2023

(1) Let $\mathbb{S}_n$ denote the symmetric group on n elements and $\{1, -1\}$ the multiplicative group of order 2.

(i) Show that $\text{sign} : \mathbb{S}_n \to \{1, -1\}$ is the only nontrivial homomorphism from $\mathbb{S}_n$ to $\{1, -1\}$.

(ii) Show that $\ker\{\text{sign}\}$, i.e. the alternating group A_n, is the only subgroup of $\mathbb{S}_n$ of index 2.

(2) Let $h \colon \mathbb{Z}^3 \to \mathbb{Z}^3$ be the homomorphism

$$h(x, y, z) = (2x + y - 16z, 8x + 4y + 2z, 2x + y - 22z).$$

Compute the quotient group $\mathbb{Z}^3/h(\mathbb{Z}^3)$ as a direct sum of cyclic groups.

(3) Let G be a group and $H < G$, $K < G$ two subgroups. Define

$$HK \stackrel{\text{def}}{=} \{x \in G \mid \text{ there exist } h \in H \text{ and } k \in K \text{ satisfying } hk = x\}$$

(i) Prove that HK is a subgroup of G if and only if $HK = KH$.

(ii) Give an example, with complete explanation, of a group G and two subgroups $H < G$, $K < G$ so that HK is not a subgroup of G.

(4) Let $T : \mathbb{C}^k \to \mathbb{C}^k$ be a linear transformation and let J be an $n \times n$ Jordan block of T with eigenvalue λ. Find the Jordan form of J^2. (Hint: consider the cases (i) $\lambda = 0$ and (ii) $\lambda \neq 0$.)

(5) Let $\mathbb{F}_3$ denote the field of 3 elements, that is, $\mathbb{F}_3 = (\mathbb{Z}/3\mathbb{Z}, +, \cdot)$.

(i) How many *distinct* 1 dimensional subspaces does a vector space of dimension 3 over $\mathbb{F}_3$ have? Give a complete explanation of how you reached your answer.

(ii) How many *distinct* 2 dimensional subspaces does a vector space of dimension 3 over $\mathbb{F}_3$ have? Give a complete explanation of how you reached your answer.

(6) Let R be the subring of $\mathbb{Q}$ consisting of all rational numbers of the form a/b where a and b are integers, and b is relatively prime to 35. Show that R has exactly two maximal ideals and describe these two maximal ideals.

(7) Let $f : R \to S$ be a surjective ring homomorphism. Prove the following statements:

(i) If $Q \subseteq S$ is a prime ideal then $f^{-1}(Q)$ is a prime ideal of Q containing $\ker f$.

(ii) If $P \subseteq R$ is a prime ideal such that $\ker f \subseteq P$ then $f(P)$ is a prime ideal of S.

(iii) There is a bijection between prime ideals of R containing $\ker f$ and prime ideals of S.

(8) Let F be an extension field of K, and $u \in F$ algebraic over K of odd degree $2n + 1$. Show that

(a) u^2 is algebraic over K of degree $2n + 1$, and

(b) $K(u) = K(u^2)$.

(9) Show that these two fields are equal:

$$\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$