Tier 1 Exam—Algebra

January 2024

You may answer as many questions as you like. All questions count equally. Show your computations and justify your answers; a correct answer without a correct proof earns little credit.

Write your solution of each problem on a separate page. Write the problem number on each page. At the end, of the exam, assemble your solutions with the problems in increasing order. You have 4 hours.

1. Let A be a 2×2 complex matrix and suppose that $\operatorname{tr} A = 2$ and $A^r = I$ for some positive integer r, where I is the 2×2 identity matrix. Prove that $A = I$.

2. a) Let $\mathbb{F}_q$ denote the field with q elements, and let V be an n-dimensional vector space over $\mathbb{F}_q$. How many different ordered bases does V have?

   b) Assume F_q and V are as above, with $n \geq 2$. How many distinct 2-dimensional vector subspaces does V have?

3. a) Let A and B be two $n \times n$ complex matrices that commute with each other. Let $\lambda \in \mathbb{C}$, and let $W = \{w \in \mathbb{C}^n : \ Aw = \lambda w\}$. Prove that for all $w \in W$, $Bw \in W$.

   b) Prove that if A and B are $n \times n$ complex matrices that commute with each other, they have a common eigenvector.

4. a) How many non-isomorphic abelian groups of order 24 are there? Give a representative of each isomorphism class.

   b) Find two non-isomorphic non-abelian groups of order 24 and show that they are not isomorphic to each other.

5. Find an element $\sigma \in S_{12}$ of the largest possible order (where S_{12} denotes the group of permutations of 12 objects) and prove that its order is indeed the largest possible.

6. a) Assume that a group G has exactly three subgroups, $\{e\}$, G itself, and one more. Prove that G is cyclic. What can you say about its order? (Note that in all parts of this problem, subgroups are considered different from each other if their elements are distinct subsets of the elements of G.)

   b) Show that if a group G contains an element of infinite order, G has infinitely many subgroups.

   c) Prove that if a group G has a finite number of subgroups then G must have finite order.

7. Let K be an algebraic extension of a field F, and let R be a *subring* of K containing F. Prove that R is a subfield of K containing F. (Caution: the degree $[K : F]$ may be infinite.)

8. Show that the group $R^\times$ of invertible elements in the ring $R = \mathbb{Z}/616\mathbb{Z}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

9. a) For which $a \in \mathbb{Z}/9\mathbb{Z}$ does the equation $z^2 = a$ have a solution in $\mathbb{Z}/9\mathbb{Z}$? Justify your answers, e.g. by checking explicitly.

   b) For which $a \in \mathbb{Z}/9\mathbb{Z}$ does the equation $z^3 = a$ have a solution in $\mathbb{Z}/9\mathbb{Z}$? Justify your answers, e.g. by checking explicitly.

   c) Count (with justification) the number of ring homomorphisms from $\mathbb{Z}[x, y]/(y^2 - x^3)$ to $\mathbb{Z}/9\mathbb{Z}$ (the convention being that a ring homomorphism $f : R \to S$ maps 1_R to 1_S).

10. Let m, n be rational numbers, neither of which is a perfect square. What are the possible values of the degree of the extension $\mathbb{Q}(\sqrt{m}, \sqrt{n})$ over $\mathbb{Q}$? Give examples of m and n where each of the possible degrees occurs, and prove that no other degree is possible.