1. Let $G$ be a finite group and $H$ a normal subgroup of $G$ of order 5. Prove that if $H$ contains an element not in the center of $G$, then $G$ has an element of order 2.

2. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. We say a subset $S$ of the additive group $G$ generates $G$ if every element of $G$ can be expressed as a sum of elements of $S$ with repetition allowed. Prove that there is no two-element subset $S = \{x, y\}$ of $G$ which generates $G$.

3. Prove that every homomorphism from any symmetric group to $\mathbb{Z}/3\mathbb{Z}$ is trivial.

4. Let $A$ and $B$ be $n \times n$ matrices over any field $F$.
   (a) Prove $AB$ and $BA$ have the same trace.
   (b) Prove that if $F = \mathbb{C}$, then $AB - BA = I$ is impossible.

5. If $A$ is a real $n \times n$ matrix and $A^2 = -I$, what are the possible eigenvalues of $A$? If $A$ is such a matrix, show that $n$ must be even. For each even $n$, give an explicit example of such a real matrix $A$.

6. Let $A$ be a complex square matrix and $x$ and $y$ be column vectors. If $Ax = \lambda_1 x$ and $A^T y = \lambda_2 y$ with $\lambda_1 \neq \lambda_2$, show that $x^T y = 0$. ($B^T$ denotes the transpose of matrix $B$.)

7. Show that if $A$ is a real $m \times n$ matrix with linearly independent columns then $A^T A$ is invertible.

8. Consider the subring $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \bigg| a, b, d \in \mathbb{C} \right\}$ of $M_2(\mathbb{C})$.
   (a) Define a surjective ring homomorphism $\phi : R \to \mathbb{C} \times \mathbb{C}$ and prove that it is indeed a ring homomorphism.
   (b) Find two distinct maximal two-sided ideals of $R$.

9. (a) Show that $\mathbb{Q}[x]/(x^2 + x + 1)$ is a field. Find the multiplicative inverse of the class represented by $x + 1$ in this field.
   (b) Show that $\mathbb{C}[x]/(x^2 + x + 1)$ is not a field.

10. Let $F$ be a field and let $F[[x]]$ be the ring of formal power series. In other words, elements of $F[[x]]$ are infinite sums $\sum_{n=0}^{\infty} a_n x^n$ which add and multiply like polynomials. In particular, the $x^n$ coefficient of $\sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} b_j x^j$ is $\sum_{i+j=n} a_i b_j$.
    (a) Find all units in $F[[x]]$.
    (b) Show that every ideal in $F[[x]]$ is principal.