Tier 1 Algebra Exam

August 2023

You may answer as many questions as you like and all questions count equally. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. Write the problem number on each page. At the end, assemble your solutions with the problems in increasing order. You have four hours.

1. Let G be a finite group and H a normal subgroup of G of order 5. Prove that if H contains an element not in the center of G, then G has an element of order 2.

2. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. We say a subset S of the additive group G *generates* G if every element of G can be expressed as a sum of elements of S with repetition allowed. Prove that there is no two-element subset $S = \{x, y\}$ of G which generates G.

3. Prove that every homomorphism from any symmetric group to $\mathbb{Z}/3\mathbb{Z}$ is trivial.

4. Let A and B be $n \times n$ matrices over any field F

(a) Prove AB and BA have the same trace.

(b) Prove that if $F = \mathbb{C}$, then $AB - BA = I$ is impossible.

5. If A is a real $n \times n$ matrix and $A^2 = -I$, what are the possible eigenvalues of A? If A is such a matrix, show that n must be even. For each even n, give an explicit example of such a real matrix A.

6. Let A be a complex square matrix and x and y be column vectors. If $Ax = \lambda_1 x$ and $A^T y = \lambda_2 y$ with $\lambda_1 \neq \lambda_2$, show that $x^T y = 0$. (B^T denotes the transpose of matrix B.)

7. Show that if A is a real $m \times n$ matrix with linearly independent columns then $A^T A$ is invertible.

8. Consider the subring $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{C} \right\}$ of $M_2(\mathbb{C})$.

(a) Define a surjective ring homomorphism $\phi : R \to \mathbb{C} \times \mathbb{C}$ and prove that it is indeed a ring homomorphism.

(b) Find two distinct maximal two-sided ideals of R.

9. (a) Show that $\mathbb{Q}[x]/(x^2 + x + 1)$ is a field. Find the multiplicative inverse of the class represented by $x + 1$ in this field.

(b) Show that $\mathbb{C}[x]/(x^2 + x + 1)$ is not a field.

10. Let F be a field and let $F[[x]]$ be the ring of formal power series. In other words, elements of $F[[x]]$ are infinite sums $\sum_{n=0}^{\infty} a_n x^n$ which add and multiply like polynomials. In particular, the x^n coefficient of $\sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} b_j x^j$ is $\sum_{i+j=n} a_i b_j$.

(a) Find all units in $F[[x]]$.

(b) Show that every ideal in $F[[x]]$ is principal.