Tier 1 Algebra Exam

August 2022

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

$\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the field of rational numbers, respectively.

1. Let A be the following matrix over $\mathbb{Q}$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

   (a) Find the characteristic polynomial of A.

   (b) Find the minimal polynomial of A.

2. Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix.

   (a) Prove that
$$\text{nullity}(A) \geq \text{nullity}(AB) - \text{nullity}(B).$$

   (b) Use part (a) to prove Sylvester's rank inequality:
$$\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n.$$

3. The inequality in problem 2 (b) implies that for a 5×5 matrix A of rank 3, one has $\text{rank}(A^2) \geq 1$. Find all Jordan canonical forms of complex 5×5 matrices A with $\text{rank}(A) = 3$ and $\text{rank}(A^2) = 1$.

4. (a) Let A be an invertible $n \times n$ matrix with integer entries. Prove that
$$A^{-1} \text{ has all integer entries if and only if } det(A) = \pm 1.$$

   (b) Let R be a commutative ring with unit, and let $M_n(R)$ be the ring of $n \times n$ matrices with entries in R. Fill in the blank along the lines of part (a):

   A has a two-sided inverse in $M_n(R)$ if and only if____________________

   Your answer should mention $det(A)$. You need not prove your answer.

5. Let G be a group G of order 60 which contains a subgroup H of order 12.

   (a) Find a non-trivial homomorphism φ from G to the group of permutations of the cosets of H.

   (b) Now assume that G is simple. Prove that φ is a one-to-one function whose image is the alternating group A_5.

6. Let G be a group, and denote by Z its center.

   (a) Prove that if G/Z is cyclic, then G is abelian.

   (b) Prove that if G is non-abelian, then $[G : Z] \geq 4$.

7. Let G be finite group. Let

$$C = \{(x, y) \in G \times G : xy = yx\}.$$

   Show that $|C| = h|G|$, where h is the number of conjugacy classes in G.

8. Let $\alpha = \sqrt{6 - 3\sqrt{2}}$, and let $K = \mathbb{Q}(\alpha)$.

   (a) Find a polynomial $p(x) \in \mathbb{Q}[x]$ such that $K \cong \mathbb{Q}[x]/(p(x))$.

   (b) Show that $\sqrt{2} \in K$, and find a basis for K as a vector space over $\mathbb{Q}(\sqrt{2})$.

9. An idempotent element of a ring is an element x such that $x^2 = x$. Find an idempotent in $\mathbb{Z}/2022\mathbb{Z}$ other than 0 or 1, using the prime factorization $2022 = 2 \times 3 \times 337$.

10. Let $\omega = \sqrt{2}$, and let $R = \mathbb{Z}[\omega]$. Consider the function $N : R \setminus 0 \to \mathbb{N}$ given by

$$N(a + b\omega) = |a^2 - 2b^2|.$$

   You may use without proof that N is a multiplicative function: $N(\alpha\beta) = N(\alpha)N(\beta)$.

   (a) Show that R is a Euclidean domain with norm function N.

   (b) Show that the units in R are precisely the elements α with $N(\alpha) = 1$.

   (c) Show that a prime number p is a prime element in R if and only if 2 is *not* a square modulo p (i.e., there is no integer k such that $k^2 \equiv 2 \bmod p$).