## Algebra Tier 1

## January 2022

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

 $\mathbf{Q}$ ,  $\mathbf{R}$ , and  $\mathbf{C}$  denotes the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. Find the Jordan canonical form of the complex matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Problem 2.** Let A and B be  $n \times n$  complex matrices such that AB = BA. Prove that A and B have a common eigenvector: there exists  $0 \neq X \in \mathbb{C}^n$  such that

$$AX = \lambda X$$
 and  $BX = \mu X$ 

for some  $\lambda, \mu \in \mathbf{C}$ .

**Problem 3.** Let  $\mathbf{F}_7$  be the field with 7 elements, and consider the special linear group

$$SL_3(\mathbf{F}_7) = \{ A \in M_{3 \times 3}(\mathbf{F}_7) \mid \det A = 1 \}.$$

What is its order  $|SL_3(\mathbf{F}_7)|$ ?

Problem 4. Give an example of 3 pairwise nonisomorphic groups of order 24.

**Problem 5.** Give an example of a group G that contains a conjugacy class with 5 elements.

**Problem 6.** Let G and H be finite groups of orders |G| = 34, |H| = 100. Prove that there exists a homomorphism

$$\phi: G \to H$$

such that  $\ker(\phi) \neq G$ .

**Problem 7.** Let  $\mathbf{F}_{11}$  be the field with 11 elements. Consider the quotient rings  $A = \mathbf{F}_{11}[x]/(x^2-2)$ and  $B = \mathbf{F}_{11}[x]/(x^2-5)$ . Is there a ring homomorphism  $\phi : A \to B$ ? Is there a ring homomorphism  $: B \to A$ ?

**Problem 8.** Give an example of two distinct polynomials  $p(x) = x^2 + ax + b$  and  $q(x) = x^2 + cx + d$ in  $\mathbf{Q}[x]$  such that p(x) and q(x) are irreducible, and the quotient fields  $\mathbf{Q}[x]/(p(x))$  and  $\mathbf{Q}[x]/(q(x))$ are isomorphic.

**Problem 9.** Consider the polynomial ring  $\mathbf{C}[x, y]$  with the ideal  $I = (x^2 + y^2, x^2 - y^2 - 1)$ . Prove that the quotient ring  $\mathbf{C}[x, y]/I$  is not a field.

**Problem 10.** Prove that the quotient ring  $K = \mathbf{Q}[x]/(x^8 - 5)$  is a field. Then show that the polynomial  $t^3 - 7$  is irreducible in the polynomial ring K[t].