## Algebra Tier 1

## January 2022

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.
$\mathbf{Q}, \mathbf{R}$, and $\mathbf{C}$ denotes the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. Find the Jordan canonical form of the complex matrix

$$
A=\left[\begin{array}{lllll}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Problem 2. Let $A$ and $B$ be $n \times n$ complex matrices such that $A B=B A$. Prove that $A$ and $B$ have a common eigenvector: there exists $0 \neq X \in \mathbf{C}^{n}$ such that

$$
A X=\lambda X \quad \text { and } \quad B X=\mu X
$$

for some $\lambda, \mu \in \mathbf{C}$.
Problem 3. Let $\mathbf{F}_{7}$ be the field with 7 elements, and consider the special linear group

$$
S L_{3}\left(\mathbf{F}_{7}\right)=\left\{A \in M_{3 \times 3}\left(\mathbf{F}_{7}\right) \mid \operatorname{det} A=1\right\}
$$

What is its order $\left|S L_{3}\left(\mathbf{F}_{7}\right)\right|$ ?
Problem 4. Give an example of 3 pairwise nonisomorphic groups of order 24.
Problem 5. Give an example of a group $G$ that contains a conjugacy class with 5 elements.
Problem 6. Let $G$ and $H$ be finite groups of orders $|G|=34,|H|=100$. Prove that there exists a homomorphism

$$
\phi: G \rightarrow H
$$

such that $\operatorname{ker}(\phi) \neq G$.
Problem 7. Let $\mathbf{F}_{11}$ be the field with 11 elements. Consider the quotient rings $A=\mathbf{F}_{11}[x] /\left(x^{2}-2\right)$ and $B=\mathbf{F}_{11}[x] /\left(x^{2}-5\right)$. Is there a ring homomorphism $\phi: A \rightarrow B$ ? Is there a ring homomorphism $: B \rightarrow A$ ?

Problem 8. Give an example of two distinct polynomials $p(x)=x^{2}+a x+b$ and $q(x)=x^{2}+c x+d$ in $\mathbf{Q}[x]$ such that $p(x)$ and $q(x)$ are irreducible, and the quotient fields $\mathbf{Q}[x] /(p(x))$ and $\mathbf{Q}[x] /(q(x))$ are isomorphic.

Problem 9. Consider the polynomial ring $\mathbf{C}[x, y]$ with the ideal $I=\left(x^{2}+y^{2}, x^{2}-y^{2}-1\right)$. Prove that the quotient ring $\mathbf{C}[x, y] / I$ is not a field.

Problem 10. Prove that the quotient ring $K=\mathbf{Q}[x] /\left(x^{8}-5\right)$ is a field. Then show that the polynomial $t^{3}-7$ is irreducible in the polymonial ring $K[t]$.

