Algebra Tier I exam – January 2021

Work all problems. They all count equally. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Begin the solution of each problem on a separate page. You have 4 hours.

1. Let I be an ideal in a commutative ring R. Set

$$J = \{x \in R\colon \text{ there exists } n > 0 \text{ so that } x^n \in I\}.$$

(a) Show that J is an ideal in R and $J \supset I$.

(b) What is J if I is a prime ideal? Prove your answer.

2. Let R be a commutative ring and let S be the ring that is $R \oplus R \oplus R$ as an abelian group with multiplication

$$(r_1, r_2, r_3) \cdot (r_4, r_5, r_6) := (r_1 r_4, r_1 r_5 + r_2 r_4, r_1 r_6 + r_3 r_4).$$

(You may assume without proof that S is indeed a ring.)

(a) What is the unit in the ring S?

(b) Which elements $(r_1, r_2, r_3) \in S$ are invertible?

(c) For any invertible element $(r_1, r_2, r_3) \in S$, give a formula for its inverse.

3. A ring is simple if every 2-sided ideal in this ring is either zero or the whole ring. Prove that for all $n \geq 1$, the ring of $(n \times n)$-matrices over a field F is simple.

4. Let F be a finite field of characteristic p. Let q be the number of elements of F. Prove that:

(a) the minimal subfield of F is isomorphic to $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$;

(b) $q = p^n$ for some integer $n \geq 1$;

(c) $a^q = a$ for all $a \in F$.

5. Prove that the quotient group G/Z of a non-abelian group G by its center $Z = Z(G)$ cannot be a cyclic group.

6. Find the order of the group $\mathrm{Aut}(\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}_9)))$, where $\mathbb{Z}_9 = \mathbb{Z}/9\mathbb{Z}$ and $\mathrm{Aut}(G)$ is the group of automorphisms of the group G.

7. Let p be a prime number and let $G = \mathrm{GL}_2(\mathbb{Z}_p)$ be the group of invertible 2-by-2 matrices with entries in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

(a) Find subgroups of G of order $p - 1$, p, $(p - 1)^2$, and $p(p - 1)$.

(b) Show that G has no subgroups of order p^2.

8. Let V be a finite-dimensional vector space, and let U_1, U_2, U_3 be subspaces of V.

(a) Prove that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

(b) Prove that

$$\begin{aligned}\dim(U_1 + U_2 + U_3) \leq{}& \dim U_1 + \dim U_2 + \dim U_3 \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

(c) Give an example of V, U_1, U_2, U_3 such that the inequality in (b) is not an equality.

9. Consider the $n \times n$ matrix over $\mathbb{R}$:

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

(a) Calculate its characteristic polynomial.

(b) Find an eigenvalue of this matrix, and a corresponding eigenvector.

(c) What is the multiplicity of your eigenvalue?

(d) Is this matrix invertible? Why?

10. Let A be an $m \times n$ matrix, and let B be an $n \times m$ matrix over a field.

(a) If $A \cdot B = I_m$, then what is $\text{rank}(B \cdot A)$? Prove your answer.

(b) Suppose that $A \cdot B = I_m$. Give a necessary and sufficient condition on n, m for $B \cdot A = I_n$.

(c) Give an example of m, n, A, B such that

$$\text{rank}(A \cdot B) < \text{rank}(A) = \text{rank}(B) = m \leq n.$$