Tier 1 Algebra exam – August 2021

All problems carry equal weight. All your answers should be justified. A correct answer without a correct proof earns little credit. Begin the solution of each problem on a new sheet of paper.

1. Let V be a finite vector space over the field $\mathbb{R}$ of real numbers.

   Suppose that $L : V \to V$ is an $\mathbb{R}$-linear map whose minimal polynomial (that is, the lowest degree monic polynomial which anihilates L) equals

   $$t^3 - 2t^2 + t - 2 \ .$$

   Prove that there is a non-zero subspace $W \subset V$ such that $L^4|_W = \mathrm{id}_W$, i.e., the restriction of $L^4 = L \circ L \circ L \circ L$ to W is the identity map.

2. (a) (5 pts) Let V be a finite-dimensional vector space over a field k which is not assumed to be algebraically closed. Let $L : V \to V$ be an endomorphism of V, and $v_1, \ldots, v_n$ eigenvectors of L with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. Assume that $\lambda_i \neq \lambda_j$ for all $i \neq j$. Prove that $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors.

   (b) (5 pts) Let k be a field and $a, b, c \in k$ fixed but arbitrary elements. Find the dimension of the kernel of the following map

   $$k^3 \longrightarrow k^3 \ , \quad (x_1, x_2, x_3) \mapsto (x_1 + ax_2, x_2 + bx_3, x_3 + cx_1) \ .$$

   Remark. Your answer will depend on a, b, c.

3. Let V be a vector space over a field k of dimension $n \geq 1$, let $\phi : V \to k$ be a *non-zero* linear function, and let $a \in V$ be a fixed *non-zero* vector. Consider the endomorphism $L_a : V \to V$ defined by $L_a(v) = v + \phi(v)a$.

   (i) Give the characteristic polynomial $\chi_{L_a}(t)$ of L_a.

   (ii) Determine the minimal polynomial of L_a (your answer may depend on $\phi(a)$).

   (iii) Identify the set of vectors $a \in V$ for which L_a is diagonalizable.

4. Let $L : \mathbb{C}^6 \to \mathbb{C}^6$ be such that $L^6 = 0$ and $\mathrm{rank}(L^2) = 2$. Describe all possibilities for the Jordan canonical form of L.

5. Give explicit examples/descriptions of

(a) two non-isomorphic *abelian* groups A, B of order 32, and

(b) two non-isomorphic *non-abelian* groups G, H of order 32,

with complete arguments that A is not isomorphic to B and G is not isomorphic to H.

6. Given a *surjective* homomorphism of groups $\phi : G \to H$, define

$$\Gamma(\phi) = \{(g, \phi(g)) \mid g \in G\} \subset G \times H.$$

Prove that $\Gamma(\phi)$ is a subgroup of $G \times H$ and that $\Gamma(\phi)$ is a normal subgroup if and only if H is abelian.

7. Let G be a group of order p^4 for a prime number p with $|Z(G)| = p^2$. Calculate the number of conjugacy classes in G as a function of p.

8. Set $\omega = \sqrt{-6}$ and let $A = \mathbb{Z}[\omega] = \{a + b\omega \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$.

(a) Define a surjective ring homomorphism $f : A \to \mathbb{Z}/5\mathbb{Z}$.

(b) Let I be the kernel of f. Show that I is not a principal ideal, i.e., cannot be generated by one element.

Hint. For part (b) you may use the function $N(a+b\omega) = a^2 + 6b^2$, which is just the square of the complex absolute value.

9. (a) Let E be a finite field extension of F that is generated over F by a subset $S = \{\alpha_1, \ldots, \alpha_n\}$ of E.

(a) Suppose $\alpha^2 \in F$ for all $\alpha \in S$. Show that the degree of the extension, $[E : F]$ (i.e. the dimension of E as an F vector space), is a power of 2.

(b) Suppose $\alpha^3 \in F$ for all $\alpha \in S$. Give an example for which $[E : F]$ is not a power of 3.

10. Let $\alpha = \sqrt[3]{2}$, $\beta = \sqrt[5]{2}$ be the positive third and fifth root of 2 in $\mathbb{R}$, respectively. Set $F = \mathbb{Q}(\alpha, \beta)$.

(a) Find $[F : \mathbb{Q}]$.

(b) Set $\gamma := \frac{\alpha^2}{\beta^3} \in F$ and show that $F = \mathbb{Q}(\gamma)$.