1. Let $V$ be a finite vector space over the field $\mathbb{R}$ of real numbers. Suppose that $L : V \to V$ is an $\mathbb{R}$-linear map whose minimal polynomial (that is, the lowest degree monic polynomial which annihilates $L$) equals $t^3 - 2t^2 + t - 2$.

Prove that there is a non-zero subspace $W \subset V$ such that $L^4|_W = \text{id}_W$, i.e., the restriction of $L^4 = L \circ L \circ L \circ L$ to $W$ is the identity map.

2. (a) (5 pts) Let $V$ be a finite-dimensional vector space over a field $k$ which is not assumed to be algebraically closed. Let $L : V \to V$ be an endomorphism of $V$, and $v_1, \ldots, v_n$ eigenvectors of $L$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. Assume that $\lambda_i \neq \lambda_j$ for all $i \neq j$. Prove that $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors.

(b) (5 pts) Let $k$ be a field and $a, b, c \in k$ fixed but arbitrary elements. Find the dimension of the kernel of the following map $k^3 \to k^3$, $(x_1, x_2, x_3) \mapsto (x_1 + ax_2, x_2 + bx_3, x_3 + cx_1)$.

Remark. Your answer will depend on $a, b, c$.

3. Let $V$ be a vector space over a field $k$ of dimension $n \geq 1$, let $\phi : V \to k$ be a non-zero linear function, and let $a \in V$ be a fixed non-zero vector. Consider the endomorphism $L_a : V \to V$ defined by $L_a(v) = v + \phi(v)a$.

(i) Give the characteristic polynomial $\chi_{L_a}(t)$ of $L_a$.
(ii) Determine the minimal polynomial of $L_a$ (your answer may depend on $\phi(a)$).
(iii) Identify the set of vectors $a \in V$ for which $L_a$ is diagonalizable.

4. Let $L : \mathbb{C}^6 \to \mathbb{C}^6$ be such that $L^6 = 0$ and rank($L^2$) = 2. Describe all possibilities for the Jordan canonical form of $L$.

5. Give explicit examples/descriptions of
(a) two non-isomorphic abelian groups $A, B$ of order 32, and
(b) two non-isomorphic non-abelian groups $G, H$ of order 32,

with complete arguments that $A$ is not isomorphic to $B$ and $G$ is not isomorphic to $H$.

6. Given a surjective homomorphism of groups $\phi : G \rightarrow H$, define
\[ \Gamma(\phi) = \{(g, \phi(g)) \mid g \in G\} \subset G \times H. \]
Prove that $\Gamma(\phi)$ is a subgroup of $G \times H$ and that $\Gamma(\phi)$ is a normal subgroup if and only if $H$ is abelian.

7. Let $G$ be a group of order $p^4$ for a prime number $p$ with $|Z(G)| = p^2$. Calculate the number of conjugacy classes in $G$ as a function of $p$.

8. Set $\omega = \sqrt{-6}$ and let $A = \mathbb{Z}[\omega] = \{a + b\omega \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$.
(a) Define a surjective ring homomorphism $f : A \rightarrow \mathbb{Z}/5\mathbb{Z}$.
(b) Let $I$ be the kernel of $f$. Show that $I$ is not a principal ideal, i.e., cannot be generated by one element.

*Hint.* For part (b) you may use the function $N(a + b\omega) = a^2 + 6b^2$, which is just the square of the complex absolute value.

9. (a) Let $E$ be a finite field extension of $F$ that is generated over $F$ by a subset $S = \{\alpha_1, \ldots, \alpha_n\}$ of $E$.
(a) Suppose $\alpha^2 \in F$ for all $\alpha \in S$. Show that the degree of the extension, $[E : F]$ (i.e. the dimension of $E$ as an $F$ vector space), is a power of 2.
(b) Suppose $\alpha^3 \in F$ for all $\alpha \in S$. Give an example for which $[E : F]$ is not a power of 3.

10. Let $\alpha = \sqrt[3]{2}$, $\beta = \sqrt[5]{2}$ be the positive third and fifth root of 2 in $\mathbb{R}$, respectively. Set $F = \mathbb{Q}(\alpha, \beta)$.
(a) Find $[F : \mathbb{Q}]$.
(b) Set $\gamma := \frac{\alpha^2}{\beta^3} \in F$ and show that $F = \mathbb{Q}(\gamma)$. 

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