Algebra Tier 1
January 2019

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

Q, R, and C denote the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

**Problem 1.** Find the Jordan canonical form of the complex matrix

\[
A = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

**Problem 2.** Let \( A \) be a complex square matrix such that \( A^n = I \) for some \( n \geq 1 \). Prove that \( A \) is diagonalizable.

**Problem 3.** A 5 \( \times \) 5 complex matrix \( A \) has eigenvalues 1 and 0. If the rank \( rk(A^2) = 1 \) find all possible Jordan canonical forms of \( A \).

**Problem 4.** Prove that a group of order 35 is cyclic.

**Problem 5.** Find the order of the automorphism group of the abelian group \( G = C_3 \oplus C_3 \oplus C_3 \), where \( C_3 \) is a cyclic group of order 3.

**Problem 6.** Describe the conjugacy classes in the dihedral group \( D_6 \). (\( D_6 \) has 12 elements.)

**Problem 7.** Prove that the polynomial ring \( \mathbb{Q}[x, y] \) contains an ideal \( I \) which can be generated by 3 elements, but not by 2 elements.

**Problem 8.** Give an example of a polynomial \( p(x) \in \mathbb{R}[x] \) such that the quotient ring \( \mathbb{R}[x]/(p(x)) \) is not a product of fields.

**Problem 9.** Determine which of the following ideals are prime ideals or maximal (or neither) in the polynomial ring \( \mathbb{C}[x, y] \):

\[
I_1 = (x), I_2 = (x, y^2), I_3 = (x - y, x + y), I_4 = (x - y, x^2 - y^2)
\]

**Problem 10.** Prove that the quotient ring \( K = \mathbb{Q}[x]/(x^7 - 5) \) is a field. Then show that the polynomial \( t^4 - 2 \) is irreducible in the polynomial ring \( K[t] \).