Algebra Tier 1

January 2019

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

Q, **R**, and **C** denotes the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. *Find the Jordan canonical form of the complex matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2. *Let A be a complex square matrix such that $A^n = I$ for some $n \geq 1$. Prove that A is diagonalizable*

Problem 3. *A 5×5 complex matrix A has eigenvalues 1 and 0. If the rank $rk(A^2) = 1$ find all possible Jordan canonical forms of A.*

Problem 4. *Prove that a group of order 35 is cyclic.*

Problem 5. *Find the order of the automorphism group of the abelian group $G = C_3 \oplus C_3 \oplus C_3$, where C_3 is a cyclic group of order 3.*

Problem 6. *Describe the conjugacy classes in the dihedral group D_6. (D_6 has 12 elements.)*

Problem 7. *Prove that the polynomial ring $\mathbf{Q}[x, y]$ contains an ideal I which can be generated by 3 elements, but not by 2 elements.*

Problem 8. *Give an example of a polynomial $p(x) \in \mathbf{R}[x]$ such that the quotient ring $\mathbf{R}[x]/(p(x))$ is not a product of fields.*

Problem 9. *Determine which of the following ideals are prime ideals or maximal (or neither) in the polynomial ring $\mathbf{C}[x, y]$:*
$I_1 = (x)$, $I_2 = (x, y^2)$, $I_3 = (x - y, x + y)$, $I_4 = (x - y, x^2 - y^2)$

Problem 10. *Prove that the quotient ring $K = \mathbf{Q}[x]/(x^7 - 5)$ is a field. Then show that the polynomial $t^3 - 2$ is irreducible in the polymonial ring $K[t]$.*