Tier 1 Algebra Exam

August 2018

Each problem is worth 10 points.

1. Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

2. Let S_{11} be the symmetric group in 11 letters. Find (with a proof) the smallest positive integer N such that all elements of S_{11} have order dividing N. You may leave your answer as a product of factors.

3. (a) (3 points) State the definition of a finitely generated ideal in a commutative ring R.

   (b) (7 points) Let R be the ring of continuous functions on the unit interval $[0, 1]$. Construct (with proof) an ideal in R which is <u>not</u> finitely generated.

4. Let $n \geq 1$ be an integer and let λ be a complex number. Determine the rank of the following $(n+1) \times (n+1)$ matrix. (Caution: Your answer may depend on the value of λ.)

$$\begin{pmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^n \\ 2 & 1 & \lambda & \dots & \lambda^{n-1} \\ 2 & 2 & 1 & \dots & \lambda^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 1 \end{pmatrix}$$

5. Let A and B be normal subgroups of a group G such that $A \cap B = 1$. Prove that G has a subgroup isomorphic to $A \times B$.

6. Let $F \subset K$ be fields. Suppose $\alpha, \beta \in K$ are algebraic over F. Let $f(x) \in F[x]$ be the irreducible polynomial of α over F, and let $g(x) \in F[x]$ be the irreducible polynomial of β over F. Suppose that $\deg f(x)$ and $\deg g(x)$ are relatively prime. Prove that $g(x)$ is irreducible in $F(\alpha)[x]$. (We denote by $F(\alpha)$ the subfield of K generated by F and

α. The irreducible polynomial of α over F is often called the minimal polynomial of α over F.)

7. (a) (7 points) Let V be a finite dimensional vector space over a field and let $T : V \to V$ be a linear transformation from V to itself such that $T(V) = T(T(V))$. Prove that $V = KerT \oplus T(V)$.

   (b) (3 points) Give an example showing the conclusion may not hold if $T(T(V)) \neq T(V)$.

8. Let G be a finite group whose order is a power of a prime integer. Let $Z(G)$ denote the center of G. Show that $Z(G) \neq \{e\}$.

9. Let $F \subset K$ be fields such that K is algebraic over F. Prove that if R is a *subring* of K containing F, then R is a subfield of K. (Caution: the degree $[K : F]$ may be infinite.)

10. Let R be an integral domain, and let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x]$$

be a polynomial with coefficients in R of degree $n \geq 1$. Suppose $P \subset R$ is a prime ideal such that $a_i \in P$ for all $i = 0, 1, \cdots, n-1$. Prove that if $f(x)$ is reducible, then $a_0 \in P^2$. (Note that $P^2 \subset R$ is the ideal generated by all products $\{ab : a, b \in P\}$ and that the leading coefficient of $f(x)$ is 1.)