Algebra Tier 1

January 2018

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

\( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \), and \( \mathbb{C} \) denotes the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

**Problem 1.** Find the Jordan canonical form of the complex matrix

\[
A = \begin{bmatrix} 1 & -2 & 1/2 \\ 2 & -4 & 1 \\ 3 & -6 & 3/2 \end{bmatrix}
\]

**Problem 2.** Find a matrix \( A \in M_{3\times3}(\mathbb{R}) \) of rotation of \( \mathbb{R}^3 \) by 120 degrees about the vector \([1, 1, 1]^T\).

**Problem 3.** Suppose that \( V \) is a vector space of dimension \( n \) and \( T: V \to V \) is a linear transformation having \( n \) distinct eigenvalues. If \( H \subset V \) is an \( m \) dimensional subspace of \( V \) and \( T(H) \subset H \), let \( T' \) denote the restriction of \( T \) to \( H \). Prove that \( T' \) has \( m \) distinct eigenvalues as a linear transformation \( T': H \to H \).

**Problem 4.** Let \( T_1 \) and \( T_2 \) be linear operators on \( \mathbb{C}^n \), such that \( T_1 \circ T_2 = T_2 \circ T_1 \). Prove that there exists a nonzero vector in \( \mathbb{C}^n \) which is an eigenvector for \( T_1 \) and for \( T_2 \).

**Problem 5.** Prove that a group of order 77 is cyclic.

**Problem 6.** Let \( G \) be the quotient of the abelian group \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) by the subgroup generated by the elements \((2, 1, 5)\), \((1, 2, 10)\), \((2, 1, 7)\). Write \( G \) as a direct sum of cyclic groups.

**Problem 7.** Let \( C_n \) denote the cyclic group of order \( n \). Find the order of the automorphism group of the abelian group \( G = C_5 \oplus C_5 \).

**Problem 8.** Give an example of a nontrivial group \( G \), such that its automorphism group \( \text{Aut}(G) \) contains a subgroup \( G' \), which is isomorphic to \( G \).

**Problem 9.** Let \( S \) be the set of polynomials \( p(t) \) in the ring \( \mathbb{Z}[t] \) for which \( p(1) \) is even. Is \( S \) an ideal, and if so, is it principal?

**Problem 10.** The polynomial \( p(x) = x^3 + 2x + 1 \) is irreducible in \( \mathbb{Q}[x] \), and thus \( \mathbb{Q}[x]/\langle p(x) \rangle = F \) is a field. Any element \( z \in F \) can be expressed as \( z = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \) for some \( \alpha_i \in \mathbb{Q} \). Find the values of the \( \alpha_i \) in the case that \( z = 1/(x - 1) \).