Algebra Tier 1

January 2018

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

Z, **Q**, **R**, and **C** denotes the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Problem 1. *Find the Jordan canonical form of the complex matrix*

$$A = \begin{bmatrix} 1 & -2 & 1/2 \\ 2 & -4 & 1 \\ 3 & -6 & 3/2 \end{bmatrix}$$

Problem 2. *Find a matrix* $A \in M_{3\times 3}(\mathbf{R})$ *of rotation of* $\mathbf{R}^3$ *by 120 degrees about the vector* $[1,1,1]^t$.

Problem 3. *Suppose that* V *is a vector space of dimension* n *and* $T : V \to V$ *is a linear transformation having* n *distinct eigenvalues. If* $H \subset V$ *is an* m *dimensional subspace of* V *and* $T(H) \subset H$, *let* T' *denote the restriction of* T *to* H. *Prove that* T' *has* m *distinct eigenvalues as a linear transformation* $T' : H \to H$.

Problem 4. *Let* T_1 *and* T_2 *be linear operators on* $\mathbf{C}^n$, *such that* $T_1 \circ T_2 = T_2 \circ T_1$. *Prove that there exists a nonzero vector in* $\mathbf{C}^n$ *which is an eigenvector for* T_1 *and for* T_2.

Problem 5. *Prove that a group of order 77 is cyclic.*

Problem 6. *Let* G *be the quotient of the abelian group* $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ *by the subgroup generated by the elements* $(2,1,5)$, $(1,2,10)$, $(2,1,7)$. *Write* G *as a direct sum of cyclic groups.*

Problem 7. *Let* C_n *denote the cyclic group of order* n. *Find the order of the automorphism group of the abelian group* $G = C_5 \oplus C_5$.

Problem 8. *Give an example of a nontrivial group* G, *such that its automorphism group* $Aut(G)$ *contains a subgroup* G', *which is isomorphic to* G.

Problem 9. *Let* S *be the set of polynomials* $p(t)$ *in the ring* $\mathbf{Z}[t]$ *for which* $p(1)$ *is even. Is* S *an ideal, and if so, is it principal?*

Problem 10. *The polynomial* $p(x) = x^3 + 2x + 1$ *is irreducible in* $\mathbf{Q}[x]$, *and thus* $\mathbf{Q}[x]/\langle p(x)\rangle = F$ *is a field. Any element* $z \in F$ *can be expressed as* $z = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ *for some* $\alpha_i \in \mathbf{Q}$. *Find the values of the* α_i *in the case that* $z = 1/(x-1)$.