Tier 1 Algebra Exam—August 2017

Be sure to justify all your answers. Remember to start each problem on a new sheet of paper. Each problem is worth 10 points.

(1) (a) Prove that if V is a finite dimensional vector space and $L\colon V \to V$ is linear and satisfies $L \circ L = 0$ then $\dim(\ker(L)) \geq \frac{1}{2}\dim(V)$.
(b) Give an example of $L\colon V \to V$ where $L \circ L = 0$ and $\dim(\ker(L)) = \frac{1}{2}\dim(V)$.
(c) Give an example of a linear $L'\colon V \to V$ where $\dim(\ker(L')) = \dim(\ker(L' \circ L')) = \frac{1}{2}\dim(V)$.

(2) Let A and B be two 3×3 matrices with entries in $\mathbb{R}$ so that A's minimal polynomial is $x^2 - 4$ and B's minimal polynomial is $x + 2$. Show that $A - B$ is a singular matrix. What are the possible values of $\dim(\ker(A - B))$?

(3) *True* or *False*. If a claim is true, say it is true—no proof needed. If it is false, give a counterexample and explain why it is a counterexample.
(a) A normal subgroup of a normal subgroup of a group G is normal in G.
(b) If K and L are normal subgroups of a group G, then

$$KL = \{gh \mid g \in K \text{ and } h \in L\}$$

is a normal subgroup of G.
(c) All the finitely generated subgroups of the additive group of real numbers are cyclic.
(d) For any group G, the set of elements of order two in G $\{g \in G \mid g^2 = e\}$ forms a subgroup of G.
(e) If G is finite and the center of G, $Z(G)$, satisfies $G/Z(G)$ is cyclic, then G is abelian.

(4) Consider the group $GL_2(\mathbb{C})$ of 2×2 invertible matrices with complex entries. Let

$$\alpha = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \text{ and } \beta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and let $G = \langle \alpha, \beta \rangle$ be the subgroup of $GL_2(\mathbb{C})$ which is generated by α and β.
(a) How many elements are there in G?
(b) Is G abelian?
(c) Prove that every subgroup of G is normal in G.

(5) A group G is called *Hopfian* if every onto homomorphism (epimorphism) from G to G has to be an isomorphism.
  (a) Show that finite groups are Hopfian.
  (b) Show that the finitely generated free abelian groups $\mathbb{Z}^n$ are Hopfian.
  (c) Deduce carefully that all finitely generated abelian groups are Hopfian.
  (d) Give an example of a non-Hopfian abelian group, A. Construct an epimorphism showing that A is not Hopfian.

(6) Let A be a commutative ring. Given any ideal I in A, We can define an ideal in A[x], which we denote $I[x]$, as follows: $I[x] = \{\sum_{j=0}^{n} r_j x^j \mid 0 \leq n \in \mathbb{Z},\ r_j \in I\}$. Note that $I[x]$ is the set of all polynomials in $A[x]$ all of whose coefficients are in I. One can easily show (and you may assume it is true for this problem) that $I[x]$ is an ideal in $A[x]$.
  (a) Assume that $\mathcal{M}$ is a maximal ideal in A. Is the ideal $\mathcal{M}[x]$ a maximal ideal in $A[x]$? Prove your answer.
  (b) Assume that $\mathcal{P}$ is a prime ideal in A. Is the ideal $\mathcal{P}[x]$ a prime ideal in $A[x]$? Prove your answer.

(7) Let A, B, C, and D be integral domains. If $A \times B$ is isomorphic to $C \times D$, prove that A is isomorphic to C or D.

(8) (a) Which finite fields $\mathbb{F}_q$ contain a primitive third root of unity (that is, a third root of unity other than 1)? Give a clear condition on q.
  (b) Deduce for which q the polynomial $x^2 + x + 1$ splits into linear terms in $\mathbb{F}_q[x]$. Use the quadratic formula when appropriate to find these linear terms. Explain over which fields it is not appropriate to use the quadratic formula.
  (c) Which finite fields $\mathbb{F}_q$ contain a square root of -3, and which do not? Make sure your answer covers all finite fields.

(9) Let $\mathbb{F}_2$ be the field with two elements. Consider the following rings: $\mathbb{F}_2[x]/(x^2)$, $\mathbb{F}_2[x]/(x^2+1)$, $\mathbb{F}_2[x]/(x^2+x)$, $\mathbb{F}_2[x]/(x^2+x+1)$. Which two of these rings are isomorphic to each other? Which of these rings are fields? Prove your answers.

(10) Let p and q be distinct primes.
  (a) Prove that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.
  (b) What is the degree of $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$ over $\mathbb{Q}$? Prove your answer in detail.