ALGEBRA TIER 1

Each problem is worth 10 points.

(1) Classify, up to isomorphism, all groups of order 24 which are quotient groups of $\mathbb{Z}^2$.

(2) If x, y are elements of G such that $(xy)^{11} = (yx)^{19} = 1$, then x and y are inverses of one another.

(3) Prove that for all $n \geq 3$, the symmetric group S_n contains elements x and y of order 2 such that xy is of order n.

(4) Let G be a non-trivial subgroup of the additive group $\mathbb{R}$ of real numbers such that $\{x \in G \mid -1 < x < 1\} = \{0\}$. Prove that there exists $r \geq 1$ such that $G = \{nr \mid n \in \mathbb{Z}\}$.

(5) Let V be an n-dimensional complex vector space, $T\colon V \to V$ a linear transformation, and $v \in V$ a vector. Prove that $v, Tv, T^2v, \ldots, T^n v$ spans V if and only if $v, Tv, T^2v, \ldots, T^{n-1}v$ is a basis of V.

(6) Let A and B be $m \times n$ and $n \times m$ complex matrices respectively. Show that every non-zero eigenvalue of AB is a non-zero eigenvalue of BA.

(7) If $M = (a_{i,j})_{1 \leq i,j \leq 3}$ is a 3×3 complex matrix such that M and $\bar{M} = (\overline{a_{i,j}})$ have the same characteristic polynomial, prove that M has a real eigenvalue.

(8) Let $R = \{\frac{m}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of nonnegative integers. Prove that R is a subring of $\mathbb{Q}$. For every ideal I of R, prove that there exists an ideal J of $\mathbb{Z}$ such that $I = \{\frac{m}{2^n} \mid m \in J, n \in \mathbb{N}\}$.

(9) Prove that if K is any finite extension of $\mathbb{Q}$, then there exists an integer n and a maximal ideal $\mathfrak{m}$ of the n variable polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ such that $K \cong \mathbb{Q}[x_1, \ldots, x_n]/\mathfrak{m}$.

(10) Prove that if F is a finite field whose order is a power of 3, then F contains a square root of -1 if and only if it contains a 4th root of -1.

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