Algebra Tier 1
January 2016

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

$F_2$, $F_7$, $Q$, and $R$ denotes the field with two elements, the field with seven elements, the field of rational numbers, and the field of real numbers respectively.

Problem 1. (a) Find the eigenvalues of the complex matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Find the eigenvectors of $A$.
(c) Find an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

Problem 2. Find the Jordan canonical form of the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Problem 3. Find the Jordan canonical form of the matrix

$$C = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 4. Let $V$ be a vector space of dimension 5 over $F_7$. Find the number of 2-dimensional subspaces of $V$.

Problem 5. Let $G$ be a nonabelian group of order 27. Find the class equation for $G$.

Problem 6. Prove that a group of order 51 is cyclic.

Problem 7. What is the order of the automorphism group of a cyclic group $G$ of order 100?

Problem 8. Factor the polynomial $x^8 - x$ in the ring $F_2[x]$.

Problem 9. Find the minimal polynomial of $\sqrt{2}$ over the field $Q(\sqrt{3})$.

Problem 10. Describe all prime ideals in the ring $R[x]$.