Algebra Tier 1

January 2016

All your answers should be explained and justified. A correct answer without a correct proof earns little credit. Each problem is worth 10 points. Write a solution of each problem on a separate page.

$\mathbf{F}_2$, $\mathbf{F}_7$, $\mathbf{Q}$, and $\mathbf{R}$ denotes the field with two elements, the field with seven elements, the field of rational numbers, and the field of real numbers respectively.

Problem 1. *(a) Find the eigenvalues of the complex matrix*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Find the eigenvectors of A.
(c) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Problem 2. *Find the Jordan canonical form of the matrix*

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Problem 3. *Find the Jordan canonical form of the matrix*

$$C = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 4. *Let V be a vector space of dimension 5 over $\mathbf{F}_7$. Find the number of 2-dimensional subspaces of V.*

Problem 5. *Let G be a nonabelian group of order 27. Find the class equation for G.*

Problem 6. *Prove that a group of order 51 is cyclic.*

Problem 7. *What is the order of the automorphism group of a cyclic group G of order 100?*

Problem 8. *Factor the polynomial $x^8 - x$ in the ring $\mathbf{F}_2[x]$.*

Problem 9. *Find the minimal polynomial of $\sqrt[5]{2}$ over the field $\mathbf{Q}(\sqrt[2]{3})$.*

Problem 10. *Describe all prime ideals in the ring $\mathbf{R}[x]$.*