Tier I Algebra Exam
August, 2015

• Be sure to fully justify all answers.
• Notation. The sets of integers, rational numbers, real numbers, and complex num-
bers are denoted \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \), and \( \mathbb{C} \), respectively. All rings are understood to have a 
unit and ring homomorphisms to be unit preserving.
• Scoring. Each problem is worth 10 points. Partial credit is possible. Answers may 
be graded on clarity as well as correctness.
• Please write on only one side of each sheet of paper. Begin each problem 
on a new sheet, and be sure to write a problem number and your test 
number on each sheet of paper.

1. Let \( V \) be the subspace of \( \mathbb{R}^4 \) spanned by \((1, 1, 1, 1), (2, 1, 0, 1), \) and \((0, 1, 1, 2)\). Let 
\( W \) be the subspace of \( \mathbb{R}^4 \) spanned by \((0, -1, -1, -3)\) and \((1, 0, 0, 1)\). Find a basis 
for the subspace \( V \cap W \) of \( \mathbb{R}^4 \).

2. Find a matrix \( M \) such that 
\[
M^2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}.
\]

3. Find up to similarity all matrices over a field \( \mathbb{F} \) having characteristic polynomial 
\((x - 3)^5(x - 4)^3\) and minimal polynomial \((x - 3)^2(x - 4)\).

4. Find (with justification) an element of largest order in the symmetric group \( S_{10} \).

5. In the group \( G := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \), let \( H \) be the subgroup generated by \((0, 6, 0, 0), \) 
\((0, 0, 3, 1), \) and \((0, 1, 0, 1)\). By the Fundamental Theorem of Finitely Generated 
Abelian Groups, there exists an isomorphism between \( G/H \) and a product of cyclic 
groups. Find such an isomorphism explicitly.

6. Show that a finite group which is generated by two distinct elements each of order 2 
must be isomorphic to a dihedral group.

7. Find the inverse of the unit 201 in the ring \( \mathbb{Z}/2015\mathbb{Z} \).

(Continues on other side.)
8. Consider the ring $R = \mathbb{Q}[x, \sqrt{x}, \sqrt[3]{x}, \sqrt[4]{x}, \ldots]$ consisting of finite sums of the form $\sum_{i=1}^{m} a_i x^{n_i}$ with $a_i \in \mathbb{Q}$ and $n_i$ a positive rational number whose denominator is a power of two. Show that every finitely generated ideal in $R$ is principal. Exhibit a nonprincipal ideal in $R$.

9. For a prime number $p$, let $\mathbb{F}_p$ denote the field with $p$ elements, and consider the ring $\mathbb{F}_p[x]/(x^3 - 1)$. Identify the group of units as a product of cyclic groups for:
   (a) $p = 29$
   (b) $p = 31$

   (You do not need to be explicit about the isomorphism with the product of cyclic groups.)

10. Show that the ring $A := \mathbb{C}[x, y]/(x^2 - y^2 - 1)$ is an integral domain. Further show that every nonzero prime ideal in $A$ is maximal.