Tier I Algebra Exam

August, 2015

- **Be sure to fully justify all answers.**
- **Notation.** The sets of integers, rational numbers, real numbers, and complex numbers are denoted $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. All rings are understood to have a unit and ring homomorphisms to be unit preserving.
- **Scoring.** Each problem is worth 10 points. Partial credit is possible. Answers may be graded on clarity as well as correctness.
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number and your test number on each sheet of paper.**

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1. Let V be the subspace of $\mathbb{R}^4$ spanned by $(1,1,1,1)$, $(2,1,0,1)$, and $(0,1,1,2)$. Let W be the subspace of $\mathbb{R}^4$ spanned by $(0,-1,-1,-3)$ and $(1,0,0,1)$. Find a basis for the subspace $V \cap W$ of $\mathbb{R}^4$.

2. Find a matrix M such that
$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}.$$

3. Find up to similarity all matrices over a field $\mathbb{F}$ having characteristic polynomial $(x-3)^5(x-4)^3$ and minimal polynomial $(x-3)^2(x-4)$.

4. Find (with justification) an element of largest order in the symmetric group S_{10}.

5. In the group $G := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, let H be the subgroup generated by $(0,6,0,0)$, $(0,0,3,1)$, and $(0,1,0,1)$. By the Fundamental Theorem of Finitely Generated Abelian Groups, there exists an isomorphism between G/H and a product of cyclic groups. Find such an isomorphism explicitly.

6. Show that a finite group which is generated by two distinct elements each of order 2 must be isomorphic to a dihedral group.

7. Find the inverse of the unit 201 in the ring $\mathbb{Z}/2015\mathbb{Z}$.

(Continues on other side.)

8. Consider the ring $R = \mathbb{Q}[x, \sqrt{x}, \sqrt[4]{x}, \sqrt[8]{x}, \ldots]$ consisting of finite sums of the form $\sum_{i=1}^{m} a_i x^{n_i}$ with $a_i \in \mathbb{Q}$ and n_i a positive rational number whose denominator is a power of two. Show that every finitely generated ideal in R is principal. Exhibit a nonprincipal ideal in R.

9. For a prime number p, let $\mathbb{F}_p$ denote the field with p elements, and consider the ring $\mathbb{F}_p[x]/(x^3 - 1)$. Identify the group of units as a product of cyclic groups for:

   (a) $p = 29$
   (b) $p = 31$

   (You do not need to be explicit about the isomorphism with the product of cyclic groups.)

10. Show that the ring $A := \mathbb{C}[x, y]/(x^2 - y^2 - 1)$ is an integral domain. Further show that every nonzero prime ideal in A is maximal.