ALGEBRA TIER 1

Each problem is worth 10 points.

(1) Prove or give a counterexample: every $n \times n$ complex matrix $A$ is similar to its transpose $A^t$.

(2) Let $M$ denote the $3 \times 4$ matrix

$$
\begin{pmatrix}
1 & 3 & 2 & 4 \\
2 & 4 & 3 & 5 \\
3 & 5 & 4 & 6
\end{pmatrix}.
$$

Determine with proof the dimension of the space of $3 \times 4$ matrices $N$ such that $N^t M = 0$.

(3) Let $V$ be a vector space and $V_1, V_2, V_3$ subspaces of $V$ such that $\dim(V_i) = 2$ for all $i$ and $\dim(V_i \cap V_j) = 1$ for all $i \neq j$. Prove that either $\dim(V_1 \cap V_2 \cap V_3) = 1$ or $\dim(V_1 + V_2 + V_3) = 3$.

(4) Let $V = \mathbb{C}^2$. Let $T : V \to V$ denote a $\mathbb{C}$-linear transformation with determinant $a + bi$, $a, b \in \mathbb{R}$. Prove that if we regard $V$ as a 4-dimensional real vector space, the determinant of $T$ as an $\mathbb{R}$-linear transformation of this space is $a^2 + b^2$.

(5) Let $G$ be a finite group of order $n \geq 2$.

(a) Prove that $G$ is always isomorphic to a subgroup of $\text{GL}_n(\mathbb{Z})$.

(b) Prove or disprove: $G$ is always isomorphic to a subgroup of $\text{GL}_{n-1}(\mathbb{Z})$.

(6) Prove that for any integer $n \geq 1$ and any prime $p \geq 2$, the symmetric group $S_{np}$ contains an $n$-element subset $P$ such that every non-trivial element of $S_{np}$ of order $p$ is conjugate to an element of $P$. Is there a set $P \subseteq S_{np}$ with the same property and less than $n$ elements?

(7) Let $G$ be a group which is the union of subgroups $G_1, G_2, \ldots, G_n$, $n \geq 2$. Show that there exists $k \in \{1, 2, \ldots, n\}$ such that

$$
\bigcap_{i \neq k} G_i \subseteq G_k.
$$

(8) Prove or give a counterexample:

(a) Let $f : R \to S$ be a ring homomorphism and let $I$ be a maximal ideal of $S$. Then $f^{-1}(I)$ is maximal.

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(b) Let $f: R \to S$ be a ring homomorphism and let $I$ be a maximal ideal of $S$. Then $f^{-1}(I)$ is prime.

(9) Consider the ideal $I = (2, \sqrt{-10})$ of $\mathbb{Z}[\sqrt{-10}]$.
(a) Show $I^2$ is principal.
(b) Show $I$ is not principal.
(c) Show $R/I \cong \mathbb{Z}/2\mathbb{Z}$ as abelian groups.

(10) Are $\mathbb{F}_5[x]/(x^2 + 2)$ and $\mathbb{F}_5[y]/(y^2 + y + 1)$ isomorphic rings? If so, write down an explicit isomorphism. If not, prove they are not.