Tier 1 Algebra Exam
August 2014

Be sure to justify all your answers. Remember to start each problem on a new sheet of paper. Each problem is worth 10 points.

(1) Consider the matrix

\[
M = \begin{bmatrix}
-\frac{1}{2} + i & 0 & \frac{1}{2} \\
-\frac{\sqrt{2}}{2} & 2 & \frac{\sqrt{2}}{2} \\
-\frac{1}{2} & 0 & \frac{1}{2} + i
\end{bmatrix}
\]

(where \(i = \sqrt{-1}\))

(a) Calculate the eigenvalues of \(M\).
(b) Is \(M\) diagonalizable over \(\mathbb{C}\)? Prove it is or explain why it is not.
(c) Calculate the minimal polynomial of \(M\).

(2) (a) Let \(S_n\) denote the group of permutations of the set \(\{1, 2, \ldots, n\}\). How many different subgroups of order 4 does \(S_4\) have? Justify your calculation. (Two subgroups are considered different if they are different as sets.)
(b) There is a homomorphism of \(S_4\) onto \(S_3\). (You do not need to prove that there exists such a homomorphism.) Show that there is no homomorphism of \(S_5\) onto \(S_4\).

(3) Let \(R\) be a commutative ring with unit and let \(a, b \in R\) be two elements which together generate the unit ideal. Show that \(a^2\) and \(b^2\) also generate the unit ideal together.

(4) Let \(\mathbb{F}_{p^n}\) denote the field with \(p^n\) elements and suppose that \(p^n - 1 = q_1^{a_1} \cdots q_k^{a_k}\) for distinct primes \(q_i\). Find the number of integers \(r \in \{0, 1, \cdots, p^n - 2\}\) for which the equation

\[x^r = a\]

has a solution for every \(a \in \mathbb{F}_{p^n}\).

(5) (a) Let \(G\) be a group and let \(H_1, H_2\) be normal subgroups of \(G\) for which \(H_1 \cap H_2 = \{e\}\). Assume that any \(g \in G\) can be written \(g = h_1 h_2\) with \(h_i \in H_i\). Show that \(G\) is isomorphic to the direct product \(H_1 \times H_2\).
(b) Show by giving an example that the above conclusion can be false if you only assume that one of the \(H_i\) is normal.

(6) (a) Find the degree of the splitting field of the polynomial \(x^4 + 1\) over \(\mathbb{Q}\).
(b) Find the degree of the splitting field of the polynomial $x^3 - 7$ over $\mathbb{Q}$.

(7) Show that a ring homomorphism $\phi : \mathbb{Q}[x] \rightarrow \mathbb{Q}[x]$ is an isomorphism if and only if $\phi(x) = ax + b$ for some $a, b \in \mathbb{Q}, a \neq 0$.

(8) For $V$ a vector space, $L : V \rightarrow V$ a linear map from $V$ to itself, and a positive integer $n$, let $L^n = L \circ L \circ \cdots \circ L$ ($n$ times).
(a) Give an example of a pair $V, L : V \rightarrow V$ so that $L$ is not the zero map, $L \neq Id$ and $L^2 = L$.
(b) Give an example of a pair $V, L$ so that $L \neq Id$ and $L^3 = Id$.
(c) Prove that if $V$ has finite dimension, then there exists an $N$ so that $\ker L^n = \ker L^N$ for all $n \geq N$.

(9) Consider the rings $\mathbb{F}_5[x]/(x^2)$, $\mathbb{F}_5[x]/(x^2 - 3)$, and $\mathbb{F}_5 \times \mathbb{F}_5$. Show that no two of them are isomorphic to each other.

(10) (a) Show that any ring automorphism of $\mathbb{R}$ sends every element of $\mathbb{Q}$ to itself.
(b) Show that any ring automorphism of $\mathbb{R}$ sends positive numbers to positive numbers.
(c) Deduce that $\mathbb{R}$ has no nontrivial automorphisms.