Tier 1 Algebra Exam

August 2014

Be sure to justify all your answers. Remember to start each problem on a new sheet of paper. Each problem is worth 10 points.

(1) Consider the matrix

$$M = \begin{bmatrix} -\frac{1}{2}+\mathbf{i} & 0 & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 2 & \frac{\sqrt{2}}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2}+\mathbf{i} \end{bmatrix}$$

(where $\mathbf{i} = \sqrt{-1}$)

(a) Calculate the eigenvalues of M.

(b) Is M diagonalizable over $\mathbb{C}$? Prove it is or explain why it is not.

(c) Calculate the minimal polynomial of M.

(2) (a) Let S_n denote the group of permutations of the set $\{1, 2, \ldots, n\}$. How many different subgroups of order 4 does S_4 have? Justify your calculation. (Two subgroups are considered different if they are different as sets.)

(b) There is a homomorphism of S_4 onto S_3. (You do not need to prove that there exists such a homomorphism.) Show that there is no homomorphism of S_5 onto S_4.

(3) Let R be a commutative ring with unit and let $a, b \in R$ be two elements which together generate the unit ideal. Show that a^2 and b^2 also generate the unit ideal together.

(4) Let $\mathbb{F}_{p^n}$ denote the field with p^n elements and suppose that $p^n - 1 = q_1^{a_1} \cdots q_k^{a_k}$ for distinct primes q_i. Find the number of integers $r \in \{0, 1, \cdots, p^n - 2\}$ for which the equation

$$x^r = a$$

has a solution for every $a \in \mathbb{F}_{p^n}$.

(5) (a) Let G be a group and let H_1, H_2 be normal subgroups of G for which $H_1 \cap H_2 = \{e\}$. Assume that any $g \in G$ can be written $g = h_1 h_2$ with $h_i \in H_i$. Show that G is isomorphic to the direct product $H_1 \times H_2$.

(b) Show by giving an example that the above conclusion can be false if you only assume that one of the H_i is normal.

(6) (a) Find the degree of the splitting field of the polynomial x^4+1 over $\mathbb{Q}$.

(b) Find the degree of the splitting field of the polynomial x^3-7 over $\mathbb{Q}$.

(7) Show that a ring homomorphism $\phi: \mathbb{Q}[x] \to \mathbb{Q}[x]$ is an isomorphism if and only if $\phi(x) = ax + b$ for some $a, b \in \mathbb{Q}$, $a \neq 0$.

(8) For V a vector space, $L : V \to V$ a linear map from V to itself, and a positive integer n, let $L^n = L \circ L \circ \cdots \circ L$ (n times).
(a) Give an example of a pair $V, L : V \to V$ so that L is not the zero map, $L \neq Id$ and $L^2 = L$.
(b) Give an example of a pair V, L so that $L \neq Id$ and $L^3 = Id$.
(c) Prove that if V has finite dimension, then there exists an N so that $\ker L^n = \ker L^N$ for all $n \geq N$.

(9) Consider the rings $\mathbb{F}_5[x]/(x^2)$, $\mathbb{F}_5[x]/(x^2-3)$, and $\mathbb{F}_5 \times \mathbb{F}_5$. Show that no two of them are isomorphic to each other.

(10) (a) Show that any ring automorphism of $\mathbb{R}$ sends every element of $\mathbb{Q}$ to itself.
(b) Show that any ring automorphism of $\mathbb{R}$ sends positive numbers to positive numbers.
(c) Deduce that $\mathbb{R}$ has no nontrivial automorphisms.