Tier 1 Examination - Algebra

January, 2014

Justify all answers!-except in problem 1. All rings are assumed to have an identity element. The set of real numbers is denoted by $\mathbf{R}$, the complexes by $\mathbf{C}$, and the rationals by $\mathbf{Q}$. The symmetric group is denoted S_n and the cyclic group of order n is denoted C_n.

(20)1. Find an example of each of the following (no proof necessary):

(a) A unique factorization domain that is not a principal ideal domain.

(b) A field with exactly 4 subfields, including the field itself.

(c) A subgroup of S_4 that is isomorphic to D_4, the dihedral group of order 8.

(d) A noncommutative ring R in which the only (two-sided) ideals are $\{0\}$ and R.

(10)2. Let G be a group of order n. Prove that G is isomorphic to a subgroup of $GL_n(\mathbf{C})$.

(10)3. Let G be a finite group and let p be the smallest prime dividing $|G|$. Prove that if H is a normal subgroup of G of order p, then H is contained in the center of G.

(10)4. Find, up to similarity, all the complex 4×4 matrices that satisfy the polynomial $(x-2)^2(x-3)$ but do not satisfy $(x-2)^3$.

(10)5. Find two elements A, B in $GL_2(\mathbf{C})$ each of which has finite order but such that AB does not have finite order.

(10)6. Prove that every $n \times n$ complex matrix is similar to an upper triangular matrix. (You may not use Jordan canonical form.)

(10)7. (a) Find the number of subgroups of $C_5 \times C_5$, including $\{e\}$ and the whole group.

(b) Find the order of the centralizer of the element $(1,2,3)(4,5,6)$ in S_6. (Recall that the centralizer of an element g in a group G is $\{x \in G | xg = gx\}$.)

(10)8. Let R be a commutative ring in which every nonzero ideal I, $I \neq R$, is maximal.

(a) Prove that R has at most 2 maximal ideals.

(b) Prove if R has exactly 2 maximal ideals, then there are fields F_1, F_2, such that R is isomorphic to $F_1 \oplus F_2$.

(10)9. Let K be the finite field extension of $\mathbf{Q}$ obtained by adjoining a root of the polynomial $x^6 + 3$.

(a) Prove that K contains a primitive 6-th root of unity.

(b) Prove that the polynomial $x^6 + 3$ factors into linear factors in $K[x]$.