1. (8 points) Prove Fermat’s Little Theorem: If \( p \) is a prime number and \( a \) is any integer, then \( a^p - a \) is divisible by \( p \).

2. (8 points) Compute \( A^{2013} \), where \( A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \).

3. (8 points) Construct an explicit isomorphism between \( \mathbb{Z}[\sqrt{-1}]/(7) \) and \( \mathbb{Z}[\sqrt{-2}]/(7) \), where \( (7) \) denotes the ideal generated by 7.

4. (9 points) Let \( \mathbb{Z}/n\mathbb{Z} \) be the cyclic group of order \( n > 1 \).
   
   (a) Show that the automorphism group \( Aut(\mathbb{Z}/n\mathbb{Z}) \) is abelian.
   
   (b) What is the order of the automorphism group of the finite group \( G = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z} \)?

   (c) Let \( G \) be as in part (b). Is the group \( Aut(G) \) abelian?

5. (8 points) Show that \( x^5 - (3 + i)x + 2 \) is irreducible in \( \mathbb{Z}[i][x] \).

6. (8 points) For any pair of real numbers \( a \) and \( b \), let \( M_{a,b} \) be the \( n \times n \) matrix

\[
M_{a,b} = \begin{pmatrix}
a & b & \cdots & b \\
b & a & \cdots & b \\
\vdots & \vdots & \ddots & \vdots \\
b & b & \cdots & a
\end{pmatrix}
\]

with entries \( a \) on the diagonal and \( b \) off the diagonal. Find the eigenvalues of \( M_{a,b} \) and their multiplicities.

7. (10 points) Classify (up to isomorphism) all groups of order 8. (You may use the following fact without proof: if \( g^2 = 1 \) for each element \( g \) in a group \( G \), then \( G \) is abelian.)
8. (8 points) Let $K$ be an algebraically closed field of characteristic $p > 0$, and let $q = p^n$. Show that the solutions of the equation $x^q = x$ form a subfield $F \subseteq K$.

9. (8 points) Let $M \in \mathcal{M}_n(\mathbb{C})$ be a diagonalizable complex $n \times n$ matrix such that $M$ is similar to its complex conjugate $\overline{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that $M$ is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.

10. (8 points) Let $p > 2$ be an odd prime. Let $F$ be a field with $q = p^n$ elements. How many solutions of the equation

$$x^2 - y^2 = 1$$

are there with $x, y \in F$?

11. (8 points) Let $G$ be a group. Let $t$ be the number of subgroups of $G$ that are not normal. Prove that $t \neq 1$.

12. (9 points) Let $V$ be a vector space of dimension $n$ over a finite field $F$ with $q$ elements.

(a) Find the number of 1-dimensional subspaces of $V$.

(b) Find the number of $n \times n$ invertible matrices with entries from $F$.

(c) For each $k$, $1 \leq k \leq n$, find the number of $k$-dimensional subspaces of $V$. 

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