## Tier 1 Algebra Exam <br> August, 2013

1. (8 points) Prove Fermat's Little Theorem: If $p$ is a prime number and $a$ is any integer, then $a^{p}-a$ is divisible by $p$.
2. (8 points) Compute $A^{2013}$, where $A=\left(\begin{array}{ll}-1 & 4 \\ -1 & 3\end{array}\right)$.
3. (8 points) Construct an explicit isomorphism between $\mathbb{Z}[\sqrt{-1}] /(7)$ and $\mathbb{Z}[\sqrt{-2}] /(7)$, where (7) denotes the ideal generated by 7 .
4. ( 9 points) Let $\mathbb{Z} / n \mathbb{Z}$ be the cyclic group of order $n>1$.
(a) Show that the automorphism group $A=\operatorname{Aut}(\mathbb{Z} / n \mathbb{Z})$ is abelian.
(b) What is the order of the automorphism group of the finite group $G=\mathbb{Z} / 5 \mathbb{Z} \oplus \mathbb{Z} / 25 \mathbb{Z}$ ?
(c) Let $G$ be as in part (b). Is the group $\operatorname{Aut}(G)$ abelian?
5. (8 points) Show that $x^{5}-(3+i) x+2$ is irreducible in $(\mathbb{Z}[i])[x]$.
6. (8 points) For any pair of real numbers $a$ and $b$, let $M_{a, b}$ be the $n \times n$ matrix

$$
M_{a, b}=\left(\begin{array}{cccc}
a & b & \ldots & b \\
b & a & \ldots & b \\
\vdots & \vdots & \ddots & \vdots \\
b & b & \ldots & a
\end{array}\right)
$$

with entries $a$ on the diagonal and $b$ off the diagonal. Find the eigenvalues of $M_{a, b}$ and their multiplicities.
7. (10 points) Classify (up to isomorphism) all groups of order 8. (You may use the following fact without proof: if $g^{2}=1$ for each element $g$ in a group $G$, then $G$ is abelian.)
8. (8 points) Let $K$ be an algebraically closed field of characteristic $p>0$, and let $q=p^{n}$. Show that the solutions of the equation $x^{q}=x$ form a subfield $F \subseteq K$.
9. (8 points) Let $M \in \mathcal{M}_{n}(\mathbb{C})$ be a diagonalizable complex $n \times n$ matrix such that $M$ is similar to its complex conjugate $\bar{M}$; i.e., there exists $g \in G L_{n}(\mathbb{C})$ such that $\bar{M}=g M g^{-1}$. Prove that $M$ is similar to a real matrix $M_{0} \in \mathcal{M}_{n}(\mathbb{R})$.
10. ( 8 points) Let $p>2$ be an odd prime. Let $F$ be a field with $q=p^{n}$ elements. How many solutions of the equation

$$
x^{2}-y^{2}=1
$$

are there with $x, y \in F$ ?
11. (8 points) Let $G$ be a group. Let $t$ be the number of subgroups of $G$ that are not normal. Prove that $t \neq 1$.
12. (9 points) Let $V$ be a vector space of dimension $n$ over a finite field $F$ with $q$ elements.
(a) Find the number of 1-dimensional subspaces of $V$.
(b) Find the number of $n \times n$ invertible matrices with entries from $F$.
(c) For each $k, 1 \leq k \leq n$, find the number of $k$-dimensional subspaces of $V$.

