Tier I Algebra Exam

January, 2013

- **Be sure to fully justify all answers.**
- **Notation** The sets of integers, rational numbers, real numbers, and complex numbers are denoted $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. All rings are understood to have a unit and ring homomorphisms to be unit preserving.
- **Scoring** Each problem is worth 10 points.
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number and your test number on each sheet of paper.**

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1. Give examples with *brief* justification:
   (a) A commutative ring with exactly one non-zero prime ideal.
   (b) A commutative ring with a non-zero prime ideal that is not maximal.
   (c) A UFD that is not a PID.
   (d) A 2×2 integer matrix having $1 + \sqrt{2}$ as an eigenvalue.
   (e) A polynomial of degree 4 with integer coefficients that is irreducible over the rational numbers but not irreducible when reduced mod 3, mod 5, and mod 7.

2. Let R be a commutative ring with unit and let $P < R$ be a prime ideal. Show that if R/P is a finite set, then P is a maximal ideal.

3. Find the degree of the field $\mathbb{Q}(\sqrt[4]{2})$ as an extension of the field $\mathbb{Q}(\sqrt{2})$.

4. Let $\mathbb{F}$ be a field with 8 elements and $\mathbb{E}$ a field with 32 elements. Construct a (unit preserving) homomorphism of rings $\mathbb{F} \to \mathbb{E}$ or prove that one cannot exist.

5. In $\mathbb{R}^5$, consider the subspaces
   $$V = \langle (1,2,3,3,2), (0,1,0,1,1) \rangle \quad \text{and} \quad W = \langle (0,-1,3,2,-1), (1,1,0,-1,1) \rangle,$$
   where $\langle\rangle$ indicates span. Find a basis for $V \cap W$.

6. Compute $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}^{100}$.

Test continues on other side.

7. Consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For which $n \in \mathbb{Z}$ does there exist a matrix P (with entries in $\mathbb{C}$) such that $P^n = M$?

8. Suppose that ϕ is a homomorphism from $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ to itself satisfying $\phi^5 = \mathrm{id}$ (where $\phi^5 = \phi \circ \phi \circ \phi \circ \phi \circ \phi$). Show that ϕ is the identity.

9. Consider the quotient of additive abelian groups $G = \mathbb{Q}/\mathbb{Z}$. Prove that every finite subgroup of G is cyclic.

10. Consider the order 2 subgroup $H = \{(1), (1\ 2)(3\ 4)\}$ of the symmetric group S_4.

    (a) What is the normalizer $N(H)$?

    (b) What numbers occur as orders of non-identity elements of the quotient group $N(H)/H$?

11. Classify up to isomorphism all groups with 38 elements: Give a list of non-isomorphic groups with 38 elements such that every group with 38 elements is isomorphic to one in your list. Be sure to justify that your list consists of non-isomorphic groups and that you have identified all groups with 38 elements up to isomorphism.

12. For an abelian group A and a positive integer n, consider the automorphism of A given by multiplication by n. Denote by ${}_nA$ and A/n its kernel and cokernel (quotient), respectively. Let $\phi\colon A \to B$ be a homomorphism of finite abelian groups, and assume that for all prime numbers p, ϕ induces an isomorphism ${}_pA \to {}_pB$ and an isomorphism $A/p \to B/p$. Show that ϕ is an isomorphism.