ALGEBRA TIER I
August, 2012

All answers must be justified. A correct answer without justification will receive little credit. Each problem is worth 10 points.

(1) Let \( f : \mathbb{Z}^3 \rightarrow \mathbb{Z}^4 \) be the function
\[
  f(a, b, c) = (a + b + c, a + 3b + c, a + b + 5c, 4a + 8b)
\]
(a) Prove that \( f \) is a group homomorphism.
(b) Let \( H \) denote the image of \( f \). Find an element of infinite order in \( \mathbb{Z}^4/H \).
(c) Calculate the order of the torsion subgroup of \( \mathbb{Z}^4/H \).

(2) Let \( A \) be the matrix
\[
  A = \begin{pmatrix}
    1 & \sqrt{3} \\
    -\sqrt{3} & 1
  \end{pmatrix}
\]
Calculate \([K : \mathbb{Q}]\) where \( K \) is the smallest subfield of \( \mathbb{C} \) containing \( \mathbb{Q} \) and the eigenvalues of \( A \).

(3) Let \( R = \mathbb{Z}[x]/I \), where \( I \) is the ideal generated by \( x^2 - 5x - 2 \).
Let \( S \) denote the ring of \( 2 \times 2 \) integer matrices: \( S = M_2(\mathbb{Z}) \).
Let \( B \) denote the matrix
\[
  B = \begin{pmatrix}
    1 & 2 \\
    3 & 4
  \end{pmatrix}
\]
(a) Show that there exists a unique ring homomorphism \( f : R \rightarrow S \) satisfying \( f(x + I) = B \).
(b) Let \( J \) denote the ideal in \( R \) generated by the element \( (x - 1) + I \). Is \( f(J) \) an ideal in \( S \)? Why or why not?

(4) Does there exist a non-abelian group of order 2012?

(5) Let \( n \in \{2, 3, 7\} \) and consider the ring
\[
  R_n = (\mathbb{Z}/n)[x]/(x^3 + x^2 + x + 2).
\]
For which \( n \in \{2, 3, 7\} \) (if any) is \( R_n \) a field? For which \( n \) (if any) is \( R_n \) a integral domain but not a field? For which \( n \) (if any) is \( R_n \) not an integral domain? Justify all your conclusions.
(6) Let \( R \) be the subring of \( \mathbb{R} \) given by \( R = \{ n + m\sqrt{-10} \mid m, n \in \mathbb{Z} \} \). Show that the element \( 2 - \sqrt{-10} \) is irreducible in \( R \) but not prime.

(7) Let \( G \) be a finite abelian group and let \( \phi : G \to G \) be a group homomorphism. Note that for all positive integers \( k \) the function \( \phi^k = \phi \circ \phi \circ \cdots \circ \phi \) is also a homomorphism from \( G \) to \( G \). Prove there is a positive integer \( n \) such that \( G \cong \ker(\phi^n) \times \phi^n(G) \).

(8) Let \( G \) be a group containing normal subgroups of order 3 and 5. Prove \( G \) contains an element of order 15.

(9) Let \( M \) be the following matrix:

\[
\begin{pmatrix}
1 & -1 & 2 \\
2 & -1 & 1 \\
-4 & 1 & 0 \\
3 & -2 & 3 \\
\end{pmatrix}
\]

Prove or disprove (by giving a counterexample) each of the following statements:

(a) For every \( 3 \times 4 \) complex matrix \( N \) there is a nonzero vector \( v \in \mathbb{C}^4 \) such that \( MNv = 0 \).

(b) For every \( 3 \times 4 \) complex matrix \( N \) there is a nonzero vector \( v \in \mathbb{C}^3 \) such that \( NMv = 0 \).

(10) Suppose that \( K/F \) is a finite extension of fields and \( p \) is the smallest prime dividing \( [K : F] \). Prove that for all \( \alpha \in K \), \( F(\alpha) = F(\alpha^{p^{-1}}) \).