Algebra Tier 1

January 2012

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page. The notation $\mathbb{F}_n, \mathbb{R}, \mathbb{Z}$ stands for the field with $n$ elements, the field of real numbers, and the ring of integers respectively.

**Problem 1.** Find the Jordan canonical form of the complex matrix

\[ A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

**Problem 2.** Find the matrix $A^{2011}$, where $A$ is the matrix from problem 1 above.

**Problem 3.** Find the eigenvalues and a basis for the eigenspaces of the matrix

\[ B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

**Problem 4.** Find the matrix $e^C := I + C + \frac{C^2}{2!} + \frac{C^3}{3!} + \ldots$, where

\[ C = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \]

**Problem 5.** Let $a$ and $b$ be elements of a group $G$. Prove that $ab$ and $ba$ have the same order.

**Problem 6.** Prove that if $G$ is a nonabelian group, then $G/Z(G)$ is not cyclic. Here $Z(G)$ denotes the center of $G$.

**Problem 7.** Prove that if $G$ is a finite nonabelian group of order $p^3$, where $p$ is a prime, then $Z(G) = [G,G]$, where $Z(G)$ denotes the center of $G$ and $[G,G]$ denotes the commutator subgroup of $G$.

**Problem 8.** Let $C_2$ denote a cyclic group of order 2. Determine the group $\text{Aut}(C_2 \times C_2)$, calculating its order and identifying it with a familiar group.

**Problem 9.** Find all irreducible polynomials of degree $\leq 4$ in $\mathbb{F}_2[x]$.

**Problem 10.** Find the set of polynomials in $\mathbb{F}_2[x]$ which are the minimal polynomials of elements in $\mathbb{F}_{16}$.

**Problem 11.** Prove that the rings $\mathbb{F}_{16}, \mathbb{F}_4 \times \mathbb{F}_4$, and $\mathbb{Z}/16\mathbb{Z}$ are pairwise non-isomorphic.

**Problem 12.** Find all the maximal ideals in the ring $\mathbb{R}[x]$.

**Problem 13.** Let $R$ be the ring of Gaussian integers and $I \subset R$ be an ideal. If $R/I$ has 4 elements what are the possibilities for $I$ and $R/I$?