1. For an element \( g \) of a group \( G \), define the centralizer subgroup \( C(g) = \{ h \in G \mid hg = gh \} \). What is the minimal order of the centralizer subgroup of an element of order 2 in \( S_6 \)? Explain.

2. Let \( G \) be a group and \( H_3 \) and \( H_5 \) normal subgroups of \( G \) of index 3 and 5 respectively. Prove that every element \( g \in G \) can be written in the form \( g = h_3h_5 \), with \( h_3 \in H_3 \) and \( h_5 \in H_5 \).

3. Show that every finite group whose order is at least 3 has a non-trivial automorphism.

4. The following matrix has four distinct real eigenvalues. Find their sum and their product.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
3 & 0 & 2 & 1 \\
0 & 1 & 0 & 3 \\
0 & 0 & 3 & 2
\end{pmatrix}
\]

5. Let \( T: \mathbb{R}^4 \to \mathbb{R}^2 \) be the linear transformation \( T(a, b, c, d) = (a + b - c, c + d) \). Find a basis for the null space.

6. A \( 5 \times 5 \) matrix \( A \) satisfies the equation \((A-2I)^3(A+2I)^2 = 0\). Assume that there are at least two linearly independent vectors \( v \) that satisfy \( Av = 2v \). What are the possibilities for the Jordan canonical form? List only one in each conjugacy class.

7. Prove that in a commutative ring with a finite number of elements, prime ideals are maximal.
8. Let $\mathbb{F}_4$ be the finite field with four elements. Express

\[
\mathbb{F}_4[x]/\langle x^4 + x^3 + x^2 + 1 \rangle
\]

as a product of fields. Prove your result.

9. Recall an element $r$ of a ring $R$ is a unit if there is an $s \in R$ so that $rs = 1 = sr$ and an element $r$ of a ring $R$ is nilpotent if there is a positive integer $n$ so that $r^n = 0$.

(a) Give an example of a ring $R$ and a unit $r \in R$ with $r \neq 1$.
(b) Give an example of a ring $R$ and a nilpotent element $r \in R$ with $r \neq 0$.
(c) Show that for any ring $R$ and for any element $r \in R$, that $r$ is a nilpotent element of $R$ if and only if $1 - rx$ is a unit in the polynomial ring $R[x]$.

10. Let $M_n(\mathbb{C})$ denote the vector space over $\mathbb{C}$ of all $n \times n$ complex matrices. Prove that if $M$ is a complex $n \times n$ matrix, then $C(M) = \{ A \in M_n(\mathbb{C}) \mid AM = MA \}$ is a subspace of $M_n(\mathbb{C})$ of dimension at least $n$. 
