Tier 1 Algebra Exam

January 2011

1. For an element g of a group G, define the *centralizer subgroup* $C(g) = \{h \in G \mid hg = gh\}$. What is the minimal order of the centralizer subgroup of an element of order 2 in S_6? Explain.

2. Let G be a group and H_3 and H_5 normal subgroups of G of index 3 and 5 respectively. Prove that every element $g \in G$ can be written in the form $g = h_3h_5$, with $h_3 \in H_3$ and $h_5 \in H_5$.

3. Show that every finite group whose order is at least 3 has a non-trivial automorphism.

4. The following matrix has four distinct real eigenvalues. Find their sum and their product.
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

5. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation $T(a, b, c, d) = (a + b - c, c + d)$. Find a basis for the null space.

6. A 5×5 matrix A satisfies the equation $(A-2I)^3(A+2I)^2 = 0$. Assume that there are at least two linearly independent vectors v that satisfy $Av = 2v$. What are the possibilities for the Jordan canonical form? List only one in each conjugacy class.

7. Prove that in a commutative ring with a finite number of elements, prime ideals are maximal.

8. Let $\mathbb{F}_4$ be the finite field with four elements. Express

$$\mathbb{F}_4[x]/\langle x^4 + x^3 + x^2 + 1\rangle$$

as a product of fields. Prove your result.

9. Recall an element r of a ring R is a *unit* if there is an $s \in R$ so that $rs = 1 = sr$ and an element r of a ring R is *nilpotent* if there is a positive integer n so that $r^n = 0$.

   (a) Give an example of a ring R and a unit $r \in R$ with $r \neq 1$.

   (b) Give an example of a ring R and a nilpotent element $r \in R$ with $r \neq 0$.

   (c) Show that for any ring R and for any element $r \in R$, that r is a nilpotent element of R if and only if $1 - rx$ is a unit in the polynomial ring $R[x]$.

10. Let $M_n(\mathbb{C})$ denote the vector space over $\mathbb{C}$ of all $n \times n$ complex matrices. Prove that if M is a complex $n \times n$ matrix, then $C(M) = \{A \in M_n(\mathbb{C}) \mid AM = MA\}$ is a subspace of $M_n(\mathbb{C})$ of dimension at least n.