Algebra Tier 1

January 2010

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page. The notation $\mathbf{Z}, \mathbf{Q}, \mathbf{C}$ stands for integers, rational numbers and complex numbers respectively.

Problem 1. *Let A be a $n \times n$ complex matrix which does not have eigenvalue -1. Show that the matrix $A + I_n$ is invertible. (I_n is the identity $n \times n$ matrix.)*

Problem 2. *(a) Find the eigenvalues of the complex matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

(b) Find the eigenvectors of A.
(c) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Problem 3. *Let A, B be $n \times n$ complex matrices such that $AB = BA$. Prove that there exists a vector $v \neq 0$ in $\mathbf{C}^n$ which is an eigenvector for A and for B.*

Problem 4. *Suppose that G is a group of order 60 that has 5 conjugacy classes of orders 1,15,20,12,12. Prove that G is a simple group.*

Problem 5. *Prove that any group of order 49 is abelian.*

Problem 6. *How many conjugacy classes are there in the symmetric group S_5?*

Problem 7. *Let $G = GL_2(\mathbf{F}_5)$, the group of invertible 2×2 matrices with entries in the field $\mathbf{F}_5$ with 5 elements. What is the order of G?*

Problem 8. *Let G and H be any pair of groups and let $S = Hom(G, H)$ denote the set of homomorphisms from G to H.*

a) Prove that if H is an abelian group, then the operation "+" on S given by $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ makes S into an abelian group.

b) Prove that if G is a finite cyclic group, then $Hom(G, \mathbf{Q}/\mathbf{Z})$ is isomorphic to G.

c) Find an infinite abelian group G so that $Hom(G, \mathbf{Q}/\mathbf{Z})$ is not isomorphic to G.

Problem 9. *Describe the prime ideals in the ring $\mathbf{C}[x]$.*

Problem 10. *Find the* **degree** *of the minimal polynomial of $\alpha = \sqrt{2} + \sqrt[3]{3}$ over $\mathbf{Q}$.*

Problem 11. *a) Prove that the polynomial $x^2 + x + 1$ is irreducible over the field $\mathbf{F}_2$ with two elements.*

b) Factor $x^9 - x$ into irreducible polynomials in $\mathbf{F}_3[x]$, where $\mathbf{F}_3$ is the field with three elements.

Problem 12. *Determine the following ideals in $\mathbf{Z}$ by giving generators:*

$$(2) + (3)\,, \qquad (4) + (6)\,, \qquad (2) \cap (3)\,, \qquad (4) \cap (6)$$

Problem 13. *Let $f(x) \in \mathbf{C}[x]$ be a polynomial of degree n such that f and f' (the derivative of f) have no common roots. Show that the quotient ring $\mathbf{C}[x]/(f)$ is isomorphic to $\mathbf{C} \times \ldots \times \mathbf{C}$ (n times).*