

**Tier I Algebra Exam**  
**August, 2009**

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- **Be sure to fully justify all answers.**
  - **Notation** The sets of integers, rational numbers, and real numbers are denoted  $\mathbf{Z}$ ,  $\mathbf{Q}$ , and  $\mathbf{R}$ , respectively. For a prime integer  $p$ ,  $\mathbf{Z}/p$  denotes the quotient  $\mathbf{Z}/p\mathbf{Z}$ . All rings are understood to have a unit.
  - **Scoring** Each problem is worth 10 points.
  - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
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- (1) Let  $A$  and  $B$  be finite subgroups of a group  $G$  of relatively prime orders; that is,  $\gcd(|A|, |B|) = 1$ . Prove that the function  $\phi : A \times B \rightarrow G$  defined by  $\phi(a, b) = ab$  is injective.
- (2) Find an element of largest order in the symmetric group  $S_{12}$ . Justify your answer.
- (3) (a) Let  $G$  and  $H$  be abelian groups and let  $\text{Hom}(G, H)$  denote the set of all group homomorphisms from  $G$  to  $H$ . If  $\phi, \psi \in \text{Hom}(G, H)$ , define  $\phi + \psi$  by  $(\phi + \psi)(g) = \phi(g) + \psi(g)$  for all  $g \in G$ . Prove that  $\text{Hom}(G, H)$  is an abelian group.  
(b) Let  $C$  be a cyclic group such that  $\text{Hom}(C, \mathbf{Z}/p) \cong \mathbf{Z}/p$ . What can you say about the order of  $C$ ?  
(c) Let  $G$  be a finitely generated abelian group and  $\text{Hom}(G, \mathbf{Z}/p) \cong (\mathbf{Z}/p)^n$ . What does this tell you about  $G$ ?
- (4) Let  $A, B$  be commuting  $2 \times 2$  real matrices with characteristic polynomials  $x^2 - 3x + 2$  and  $x^2 - 1$ , respectively. Show that either  $A + B$  or  $A - B$  has determinant 0.
- (5) Suppose that  $T$  is a linear transformation of a finite dimensional real vector space  $V$  having characteristic polynomial  $f(t)g(t)$  where  $f$  and  $g$  are relatively prime. Show that  $V = \text{Ker}(f(T)) \oplus \text{Ker}(g(T))$ .
- (6) Let  $T$  be a linear transformation of a finite dimensional real vector space  $V$  and assume that  $V$  is spanned by eigenvectors of  $T$ . If  $T(W) \subset W$  for some subspace  $W \subset V$ , show that  $W$  is spanned by eigenvectors. (Hint: consider the minimal polynomial of  $T$ .)

- (7) Let  $F$  denote the field with two elements and let  $E$  be an extension field of  $F$ .
- Show that if  $\alpha \in E$  satisfies  $f(\alpha) = 0$  for some  $f \in F[x]$ , then  $f(\alpha^2) = 0$ .
  - Suppose that  $\alpha \in E$  is a root of the polynomial  $f(x) = x^5 + x^2 + 1 \in F[x]$ . List all roots of  $f(x)$  in  $E$  in the form  $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4$  with each  $a_i = 0$  or  $1$ .
- (8) Let  $f(x) = x^2 + ax + b$  and  $g(x) = x^2 + cx + d$  be irreducible rational polynomials, having roots  $\alpha$  and  $\beta$ , respectively. Find necessary and sufficient conditions on the coefficients  $(a, b, c, d)$  that imply that  $\mathbf{Q}(\alpha)$  is isomorphic to  $\mathbf{Q}(\beta)$ . Prove that your conditions are both necessary and sufficient.
- (9) (a) Is the ring  $\mathbf{Z}[i]$  of Gaussian integers an integral domain? Justify your answer.
- (b) Let  $R = \mathbf{Z}[T]/\langle T^4 - 1 \rangle$ , where  $\langle T^4 - 1 \rangle$  is the ideal generated by  $T^4 - 1$ . Is  $R$  an integral domain? Justify your answer.
- (c) Show that sending  $T$  to  $i$  determines a ring homomorphism  $\psi : R \rightarrow \mathbf{Z}[i]$ . Describe  $\text{Ker } \psi$  as all elements  $a + bT + cT^2 + dT^3 \in R$  where  $a, b, c, d$  satisfy some conditions.
- (10) Let  $F^*$  denote the multiplicative group of all nonzero elements of a finite field  $F$ . Show that  $F^*$  is cyclic.