ALGEBRA TIER 1 EXAM
JANUARY, 2009

Instructions: Each problem or major part of a problem counts five points, as indicated, for a total of 90 points. Start each problem on a new sheet of paper. Unless otherwise stated you should show your work and justify your claims.

Problem 1 (10 points)**.**

(1) Prove that every subgroup of a cyclic group is cyclic.

(2) Determine, up to isomorphism, the finitely generated abelian groups with the property that every proper subgroup is cyclic.

Problem 2 (5 points)**.** *Let G be a group. Define a subgroup $H \leq G$ to be* characteristic *if for every isomorphism $\varphi : G \to G$ one has $\varphi(H) \subseteq H$. Now suppose that $H \leq G$ is a normal subgroup and $K \leq H$ is a characteristic subgroup of H. Prove that K is a normal subgroup of G.*

Problem 3 (5 points)**.** *Let G be a finite abelian group of order n, in which the group operation is written multiplicatively. Suppose that the map $f : x \mapsto x^m$ is an automorphism of G, for some positive integer m. Prove that* $\gcd(n, m) = 1$.

Problem 4 (10 points)**.** *Let V denote a finite-dimensional complex vector space, and $L(V)$ the complex vector space of linear transformations from V to itself. Given $A \in L(V)$, the function $T_A : L(V) \to L(V)$ defined by $T_A(X) = AX - XA$ for all $X \in L(V)$ is a linear transformation.*

(1) Suppose that $A, B \in L(V)$ have the same Jordan canonical form. Prove that T_A and T_B have the same Jordan canonical form.

(2) Suppose that the dimension of V is equal to 2. Prove that for every $A \in L(V)$, the rank of the transformation T_A is either 0 or 2.

Problem 5 (5 points)**.** *Let V be a real vector space and $T : V \to V$ a linear transformation. Suppose every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity.*

Problem 6 (10 points).

(1) *Let A be a 2×2 matrix with entries in the field of real numbers such that* $A^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. *Show that A is similar over the real numbers to the matrix* $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(2) *Let A be an $n \times n$ matrix with entries in the field of real numbers such that $A^2 + I_n = 0$ where I_n is the $n \times n$ identity matrix. Show that n is even and A is similar over the real numbers to the matrix* $B = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$, *where $n = 2m$.*

Problem 7 (10 points). *Let R be a commutative ring with 1 that contains exactly three ideals.*

(1) *Show that every nonzero element of R is either a unit or a zero divisor.*

(2) *Is the converse true? Justify your answer.*

Problem 8 (5 points). *For a prime integer p, let F_p denote the field of p elements. Suppose the greatest common divisor of the polynomials $f(x) = 6x^3 + 10x^2 - 10x + 16$ and $g(x) = 6x^2 + 10x - 16$ in $F_p[x]$ is 1. Find p.*

Problem 9 (10 points). *Let F_3 denote the finite field with 3 elements and $\overline{F}_3$ be its algebraic closure. Let K be the splitting field of $g(x) = x^{21} - 1$.*

(1) *Find the number of zeros of $g(x)$ in the field $\overline{F}_3$.*

(2) *(a) Find the number of elements in K. (b) What are the numbers of elements in the maximal proper subfields of K? (A subfield of K is said to be* proper *if it is not equal to K itself.)*

Problem 10 (10 points).

(1) *Suppose γ is a complex number such that γ^2 is algebraic over $\mathbb{Q}$. Show that γ is algebraic over $\mathbb{Q}$.*

(2) *Let α, β be complex numbers such that α is transcendental over $\mathbb{Q}$. Show that at least one of $\alpha - \beta$ and $\alpha\beta$ is transcendental.*

Problem 11 (10 points). *Let D be a domain. Two non-zero ideals $I, J \subset D$ are called "comaximal" if $I + J = D$ and the two ideals are called "coprime" if $I \cap J = I \cdot J$.*

(1) *Show that if the ideals $I, J \subset D$ are comaximal, then they are coprime.*

(2) *Show that if D is a principal ideal domain and the ideals $I, J \subset D$ are coprime, then they are comaximal.*