Algebra Tier 1

January 2008

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page.

Problem 1. *Find eigenvalues and the corresponding eigenvectors of the complex matrix*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 2. *Let A be a 5×5 complex matrix such that $A^3 = 0$. List all possible Jordan canonical forms of A.*

Problem 3. *Find a 5×5 matrix A with rational entries whose minimal polynomial is $(x^3+1)(x+2)^2$.*

Problem 4. *Let A be a complex $n \times n$ matrix such that $A^m = I$ for some $m \geq 1$. Prove that A is conjugate to a diagonal matrix.*

Problem 5. *Consider the group $\mathbf{R}, +$, the additive group of the real numbers.*
a) Show that any homomorphism from a finite group to $\mathbf{R}, +$ has to be the trivial homomorphism.
b) Show that any homomorphism from $\mathbf{R}, +$ to a finite group has to be the trivial homomorphism.

Problem 6. *Consider the subgroup H of the group $\mathbf{Z}/12 \times \mathbf{Z}/12$ generated by the element (a^4, a^6), where a is a generator of $\mathbf{Z}/12$.*
a) What is the order of H? List its elements.
b) How many elements are there in $(\mathbf{Z}/12 \times \mathbf{Z}/12)/H$?
c) Write $(\mathbf{Z}/12 \times \mathbf{Z}/12)/H$ as a product of cyclic groups, each of which has order equal to a power of some prime. Find a generator for each of these cyclic subgroups.

Problem 7. *Show that in a finite group of odd order every element is a square.*

Problem 8. *For each of the following subgroups of S_4 (the permutation group on four elements), say what its order is and justify your answer.*
a) The subgroup generated by $(1,2)$ and $(3,4)$.
b) The subgroup generated by $(1,2)$, $(3,4)$, and $(1,3)$.
c) The subgroup generated by $(1,2)$, $(3,4)$, and $(1,3)(2,4)$.
d) The subgroup generated by $(1,2)$ and $(1,3)$.

Problem 9. *Let R be an integral domain that contains a field K. Show that if R is a finite dimensional vector space over K, then R is a field.*

Problem 10. *Let $f(x)$ be a polynomial with coefficients from a finite field F with q elements. Show that if $f(x)$ has no roots in F, then $f(x)$ and $x^q - x$ are relatively prime.*

Problem 11. *Let α be a root of an irreducible polynomial x^3-2x+2 over $\mathbf{Q}$. Find the multiplicative inverse of $\alpha^2+\alpha+1$ in $\mathbf{Q}[\alpha]$ in the form $a + b\alpha + c\alpha^2$ with a, b, $c \in \mathbf{Q}$.*

Problem 12. *Let $f(x)$ and $g(x)$ be irreducible polynomials over $\mathbf{Q}[x]$. Let α be a root of $f(x)$ and let β be a root of $g(x)$. Show that $f(x)$ is irreducible over $\mathbf{Q}(\beta)$ if and only if $g(x)$ is irreducible over $\mathbf{Q}(\alpha)$.*