Problem 1. Find eigenvalues and the corresponding eigenvectors of the complex matrix

\[
A = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Problem 2. Let \( A \) be a 5 \times 5 complex matrix such that \( A^3 = 0 \). List all possible Jordan canonical forms of \( A \).

Problem 3. Find a 5 \times 5 matrix \( A \) with rational entries whose minimal polynomial is \((x^3 + 1)(x + 2)^2\).

Problem 4. Let \( A \) be a complex \( n \times n \) matrix such that \( A^m = I \) for some \( m \geq 1 \). Prove that \( A \) is conjugate to a diagonal matrix.

Problem 5. Consider the group \( \mathbb{R}, + \), the additive group of the real numbers.
   a) Show that any homomorphism from a finite group to \( \mathbb{R}, + \) has to be the trivial homomorphism.
   b) Show that any homomorphism from \( \mathbb{R}, + \) to a finite group has to be the trivial homomorphism.

Problem 6. Consider the subgroup \( H \) of the group \( \mathbb{Z}/12 \times \mathbb{Z}/12 \) generated by the element \((a^4, a^6)\), where \( a \) is a generator of \( \mathbb{Z}/12 \).
   a) What is the order of \( H \)? List its elements.
   b) How many elements are there in \((\mathbb{Z}/12 \times \mathbb{Z}/12)/H\)?
   c) Write \((\mathbb{Z}/12 \times \mathbb{Z}/12)/H\) as a product of cyclic groups, each of which has order equal to a power of some prime. Find a generator for each of these cyclic subgroups.

Problem 7. Show that in a finite group of odd order every element is a square.

Problem 8. For each of the following subgroups of \( S_4 \) (the permutation group on four elements), say what its order is and justify your answer.
   a) The subgroup generated by \((1,2)\) and \((3,4)\).
   b) The subgroup generated by \((1,2)\), \((3,4)\), and \((1,3)\).
   c) The subgroup generated by \((1,2)\), \((3,4)\), and \((1,3)(2,4)\).
   d) The subgroup generated by \((1,2)\) and \((1,3)\).

Problem 9. Let \( R \) be an integral domain that contains a field \( K \). Show that if \( R \) is a finite dimensional vector space over \( K \), then \( R \) is a field.

Problem 10. Let \( f(x) \) be a polynomial with coefficients from a finite field \( F \) with \( q \) elements. Show that if \( f(x) \) has no roots in \( F \), then \( f(x) \) and \( x^q - x \) are relatively prime.

Problem 11. Let \( \alpha \) be a root of an irreducible polynomial \( x^3 - 2x + 2 \) over \( \mathbb{Q} \). Find the multiplicative inverse of \( \alpha^2 + \alpha + 1 \) in \( \mathbb{Q}[\alpha] \) in the form \( a + b\alpha + c\alpha^2 \) with \( a, b, c \in \mathbb{Q} \).

Problem 12. Let \( f(x) \) and \( g(x) \) be irreducible polynomials over \( \mathbb{Q}[x] \). Let \( \alpha \) be a root of \( f(x) \) and let \( \beta \) be a root of \( g(x) \). Show that \( f(x) \) is irreducible over \( \mathbb{Q}(\beta) \) if and only if \( g(x) \) is irreducible over \( \mathbb{Q}(\alpha) \).