

**Tier I Algebra Exam**  
**August, 2007**

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- **Be sure to fully justify all answers.**
  - **Notation** The sets of integers, rational numbers, real numbers, and complex numbers are denoted  $\mathbf{Z}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ , and  $\mathbf{C}$ , respectively. All rings are understood to have a unit.
  - **Scoring** Each single part problem is worth 10 points. Each part of a multiple part problem is worth 5 points. (eg. Problem 1 is worth 10 points, Problem 2 is worth 25 points.)
  - **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
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- (1) Prove that for a group  $G$  and positive integer  $k$ , if  $G$  contains an index  $k$  subgroup, then the intersection of all index  $k$  subgroups of  $G$  is a normal subgroup.
- (2) Let  $F : G \rightarrow G$  be an endomorphism, that is, a homomorphism from the group  $G$  to itself. Let  $F^n$  denote the  $n$ -fold composition of  $F$  with itself, and let  $K_n = \text{Kernel}(F^n)$ .
- (a) Show that  $K_n \subseteq K_{n+1}$  for all  $n$ .
  - (b) Let  $F : (\mathbf{Z}/16\mathbf{Z})^3 \rightarrow (\mathbf{Z}/16\mathbf{Z})^3$  be the endomorphism defined by  $F(x, y, z) = (2z, 2x, 8y)$ . For all  $n \geq 1$ , describe  $K_n$  as a direct sum of cyclic groups.
  - (c) Show that if  $F$  is an endomorphism of the symmetric group  $S_5$ ,  $K_{n+1} = K_n$  for all  $n \geq 2$ .
  - (d) Give an example of an endomorphism  $F$  of the symmetric group  $S_5$  for which  $K_2 \neq K_1$ .
  - (e) Prove that for general  $G$  and  $F$ , if  $K_n = K_{n+1}$ , then  $K_n = K_{n+i}$  for all  $i \geq 0$ .
- (3) Let  $S = \{(x, y) \mid 23x + 31y = 1, x + y < 100\} \in \mathbf{Z}^2$ . Find the element of  $S$  for which  $x + y$  is as large as possible.
- (4) Let  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^4$  and  $S : \mathbf{R}^4 \rightarrow \mathbf{R}^1$  be linear transformations given by:
- $$T(x, y, z) = (x + 2y + z, x - y + 4z, x - y + 4z, 2x + y + 5z)$$
- $$S(x, y, z, w) = (x - y + 2z - w).$$
- Find two sets of vectors in  $\mathbf{R}^4$ ,  $\{\alpha_1, \dots, \alpha_m\}$  and  $\{\beta_1, \dots, \beta_n\}$  such that  $\{\alpha_1, \dots, \alpha_m\}$  is a basis of  $\text{Im}(T)$  and  $\{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n\}$  is a basis for  $\text{Ker}(S)$ . Justify your answer.
- (5) Let  $T_1, T_2 : \mathbf{C}^3 \rightarrow \mathbf{C}^3$  be linear transformations. Show that if both linear transformations have minimal polynomials of degrees at most 2, then there is a vector that is an eigenvector for both  $T_1$  and  $T_2$ .
- (6) Let  $R$  be a commutative ring with unity and suppose that for every  $r \in R$  there is an  $n \geq 2$  so that  $r^n = r$ . Show that every prime ideal in  $R$  is maximal.
- (7) Suppose that  $R$  is an integral domain. Is it possible that  $R$  contains additive subgroups isomorphic to  $\mathbf{Z}/p\mathbf{Z}$  and  $\mathbf{Z}/q\mathbf{Z}$  for  $p$  and  $q$  distinct primes? Justify your answer.

- (8) Prove that the polynomial  $2x^4 + x + 1 \in \mathbf{Q}[x]$  is irreducible. Justify all your work.
- (9) Let  $F_q$  denote the finite field with  $q$  elements. Show that for any  $a \in F_q$  the equation  $x^n = a$  has a solution in  $F_q$  if  $n$  is relatively prime to  $q - 1$ .
- (10) Let  $p(t) = t^3 - 2 \in \mathbf{Q}[t]$ . Let  $\alpha = \sqrt[3]{2}$  be the real root of  $p$  and let  $\beta$  be a complex root of  $p$ . Determine if  $\alpha \in \mathbf{Q}[\beta]$  and explain your answer.
- (11) Let  $d > 1$  and let  $p(x)$  and  $q(x)$  be relatively prime irreducible polynomials in  $\mathbf{Q}[x]$  of degree  $d$ . Suppose  $p(\alpha) = 0 = q(\beta)$  for some  $\alpha, \beta \in \mathbf{C}$ . It follows that  $1 \leq [\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] \leq d$ .
- (a) Find an example of a  $d, p, q, \alpha$ , and  $\beta$ , so that  $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] = 1$ .
- (b) Find an example of a  $d, p, q, \alpha$ , and  $\beta$ , so that  $[\mathbf{Q}(\alpha, \beta) : \mathbf{Q}(\alpha)] = d$ .