Tier 1 Algebra Examination, January, 2007

Important:

- Justify fully each answer unless otherwise directed.
- Notation: $\{1, 2, 3, \ldots\} = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote respectively the natural numbers, integers, rationals, reals, and complex numbers

1. **(5 each pts.)**

   (a) Give an example of groups G, H such that $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$ are finite but $\mathrm{Aut}(G \times H)$ is infinite. (Here, $\mathrm{Aut}(G)$ denotes the set of automorphisms of G.) *No proof required.*

   (b) Give an example of an ideal in $\mathbb{Z}[x]$ that is prime but not maximal. *No proof required.*

   (c) Give an example of an integral domain that is not a unique factorization domain. *No proof required.*

   (d) State Eisenstein's criterion for a polynomial $f \in \mathbb{Z}[x]$ to be irreducible over $\mathbb{Q}$. *No proof required.*

2. **(10 pts.)** Let V be the real vector space of functions on $\mathbb{R}$ spanned by the set of real-valued functions $\{e^x, xe^x, x^2e^x, e^{2x}\}$. Let $T : V \to V$ be the linear operator on V defined by $T(f) = f'$. Find (i) a Jordan canonical form of T, and (ii) a Jordan canonical basis.

3. **(10 pts.)** Let $f : V \to V$ be an endomorphism of a finite-dimensional vector space V. Show that there is a subspace U of V such that $f(U) = f(V)$ and $V = U \oplus \ker f$.

4. **(10 pts.)** Let G be a finitely generated abelian group. Prove that there are no nontrivial homomorphisms $\phi : \mathbb{Q} \to G$, where $\mathbb{Q}$ denotes the additive rationals.

5. **(10 pts.)** Let G be a simple group of order n. Let H be a subgroup of G of index k with $H \neq G$. Show that n divides $k!$.

6. **(10 pts.)** Let R be a commutative ring with unity 1. Suppose each subring of R contains 1. Prove that R is a field of nonzero characteristic.

7. **(10 pts.)** Let R be a ring with 1, let $a \in R$, and suppose $a^n = 0$ for some $n \in \mathbb{N}$. Prove that $1 + a$ is a unit of R.

8. **(10 pts.)** Let R be a commutative ring with 1. Let m be a maximal ideal of R such that $m \cdot m = 0$.

   (a) Prove that m is the only maximal ideal of R.

   (b) Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial in $R[x]$ such that $a_i \in m$ for all i and $a_0 \neq 0$. Prove that $f(x)$ is irreducible, i.e., $f(x)$ is not a product of two polynomials in $R[x]$ of degree strictly smaller than $\deg f$.

9. **(10 pts.)** Let F denote a finite field of order $2^5 = 32$. Prove that for each integer $1 \leq n < 32$ and each $a \in F$, the equation $x^n = a$ has a solution in F.

10. **(10 pts.)** Determine with proof the degree of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.