ALGEBRA TIER ONE EXAMINATION, JANUARY 2005

(1) (a) Compute the units in the ring $\mathbb{Z}_4[x]$.

(b) Find an irrreducible polynomial of degree 4 in $\mathbb{Z}_2[x]$. Justify your answer.

(2) Give examples of two 6×6 matrices A and B over $\mathbb{Q}$ with minimal polynomial $(x-2)^2(x^2+3)$ such that A is not similar to B.

(3) Find an explicit formula for the entries of the following matrix in terms of n.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$$

(4) Let G be a group containing a normal subgroup H isomorphic to D_8, the dihedral group of order 8. Prove that G must have a nontrivial center.

(5) Let ϕ be a group homomorphism from D_{18}, the dihedral group of order 18, to S_8. Prove that ϕ is not injective.

(6) Let R be a unique factorization domain.

(a) Prove that if P is a nonzero prime ideal of R then P must contain an irreducible element.

(b) A prime ideal P in an integral domain R is called minimal if $P \neq 0$ and the only prime ideals Q such that $Q \subseteq P$ are $Q = 0$ and $Q = P$. Prove that in a unique factorization domain every minimal prime is principal.

(7) Let F be a field and let $f(x) \in F[x]$ be a nonzero polynomial. Let n be the degree of $f(x)$. The quotient ring $F[x]/(f(x))$ may be viewed as an F–vector space via $\alpha(g(x)+(f(x))) = \alpha g(x)+(f(x))$ for all $\alpha \in F$ and $g(x) \in F[x]$. Prove that this vector space is finite dimensional over F and that the images of the elements $1, x, x^2, \ldots, x^{n-1}$ form a basis.

(8) Let R be a ring and let $N(R) = \{r \in R | r^k = 0 \text{ for some } k > 0\}$.

(a) Prove that if R is commutative, then $N(R)$ is an ideal in R.

(b) Give an example to show that there are noncommutative rings R in which $N(R)$ is not an ideal.

(9) Prove that for any prime p, the field $\mathbb{F}_p$ with p elements contains an element a such that $[\mathbb{F}_p(\sqrt[3]{a}) : \mathbb{F}_p] = 3$ if and only if $p-1$ is divisible by 3.

(10) For each of the following construct an example or prove that there is none.

(a) A finite field extension L/K and elements $\alpha, \beta \in L$ of degree 2 over K such that $\alpha + \beta$ is of degree 3 over K.

(b) A finite field extension L/K and elements $\gamma, \delta \in L$ of degree 3 over K such that $\gamma + \delta$ has degree 6 over K.