Tier 1 Algebra Exam—August 2004

(5 pts) 1. Let A be a 5×5 matrix with entries in $\mathbb{R}$. If $A^3 = 0$ and

$$\dim(\ker(A)) + \dim(\ker(A^2)) = 8,$$

find the Jordan canonical form of A.

(5 pts) 2. Let e_1, e_2 be two (non-zero) eigenvectors of a linear transformation $T : V \to V$, and assume $e_1 \neq -e_2$. Show that $e_1 + e_2$ is an eigenvector of T if and only if e_1, e_2 correspond to the same eigenvalue.

(6 pts) 3. Given that the characteristic polynomial of a 5×5 integer matrix is $x(x^4 + 1)$, is the matrix diagonalizable over $\mathbb{R}$? over $\mathbb{C}$? In each case, give its diagonal form if it exists or explain why it is not diagonalizable.

(5 pts) 4. List all the abelian subgroups of S_4, the permutation group on 4 elements.

(5 pts) 5. Is the group of automorphisms of an abelian group always abelian? Prove, or give a counterexample.

6. Let $\mathbb{C}^*$ denote the group whose elements are $\mathbb{C} \setminus \{0\}$, equipped with multiplication.

(4 pts) (a) Show that for any group G and any abelian group H, the group operation of H induces an operation on $\mathrm{Hom}(G, H)$ which makes it into an abelian group.

(7 pts) (b) Use the structure theorem for finite abelian groups to show that for any finite abelian group G, $\mathrm{Hom}(G, \mathbb{C}^*)$ is isomorphic to G.

(6 pts) 7. Let H, K be proper subgroups of a group G, so that neither of them is contained in the other one. In each of the following two cases, prove that $H \cap K$ must be normal in G *or* prove that it cannot be normal in G *or* give examples showing that it can either be normal or not normal. (i) H, K both normal in G. (ii) H is normal in G, but K is not.

8. Let R be an integral domain. Prove:

(5 pts) (a) An element $a \in R$ is irreducible (meaning that a is not a unit, and whenever $a = bc$ for $b, c \in R$, either b or c is a unit) if and only if Ra is maximal among the proper principal ideals of R.

(5 pts) (b) If R is a principal ideal domain, then each irreducible element in R is prime.

(5 pts) 9. Show that $\mathbb{Z}[\sqrt{-7}]$ is not a UFD.

10. Let $R = F[x]$, where F is a field. Let $f(x)$ and $g(x)$ be polynomials in R such that the degree of $f(x)$ is smaller than the degree of $g(x)$. Let $g(x) = g_1(x) \cdot g_2(x)$ for relatively prime polynomials $g_i(x)$.

(7 pts) (a) Show that there are polynomials $f_i(x)$ such that the degree of each $f_i(x)$ is smaller than the degree of $g_i(x)$ for $i = 1, 2$ and such that

$$\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} \quad .$$

(4 pts) (b) Are the $f_i(x)$ unique? Justify your answer.

(4 pts) 11. Let p be a prime and let α be algebraic over $\mathbb{Z}_p$, the field with p elements. It is given that the multiplicative order of α is k.

Show that p does not divide k.

(6 pts) 12. Find the irreducible polynomial of $\alpha = \sqrt{2} + \sqrt[3]{5}$ over $\mathbb{Q}$. Justify why the polynomial you give is α's irreducible polynomial.

13. Let p be a prime. For a natural number r, let $GF(p^r)$ denote the finite field with p^r elements.

(4 pts) (a) Prove one of the two implications in the following proposition:

Proposition. $GF(p^r) \subseteq GF(p^s)$ if and only if r divides s.

(7 pts) (b) Assuming the proposition above, prove that there are $p^{15} - p^3 - p^5 + p$ elements α in the algebraic closure of $\mathbb{Z}_p$ for which $\mathbb{Z}_p(\alpha)$ is $GF(p^{15})$.

14. Let K be an extension field of the field F, and let α be an element of K.

(5 pts) (a) If α is transcendental, show that $F(\alpha) \neq F(\alpha^2)$. Give an example to show that the reverse implication is not true.

(5 pts) (b) If α is algebraic over F and $F(\alpha) \neq F(\alpha^2)$, show that $[F(\alpha) : F]$ is even. Give an example to show that the reverse implication is not true.