ALGEBRA TIER I EXAM
January 2004

You have 25 questions. Each one is worth 4 points.

1. Let $V \neq \{0\}$ be a vector space over the real numbers $\mathbb{R}$ with the dimension $\dim(V) = n < \infty$. Let $f : V \to V$ be a linear mapping different the identical mapping id_V and different from $-\mathrm{id}_V$. Assume that f is an involution, i.e., $f \circ f = \mathrm{id}_V$.

a) Prove that the only possible eigenvalues are $\lambda = 1$ and $\lambda = -1$.

b) Let V_1 and V_{-1} be the subspace of V consisting of all the eigenvectors corresponding to the eigenvalue $\lambda = 1$ and $\lambda = -1$, respectively. Prove that $V_1 \neq \{0\}$, $V_{-1} \neq \{0\}$, $V_{-1} \cap V_1 = \{0\}$, and $V_{-1} + V_1 = V$.

c) Prove that there exists a basis for V such that the matrix representation F for f takes the form of a 2×2 block matrix

$$F = \begin{pmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & -I_q \end{pmatrix},$$

where I_p, 1_q, $0_{p\times q}$, and $0_{q\times p}$, denote the $p \times p$ unit matrix, the $q \times q$ unit matrix, the $p \times q$ zero matrix, and the $q \times p$ zero matrix, respectively, and where $p + q = n$, with $p > 0$ and $q > 0$.

2. Let $V \neq \{0\}$ be a vector space over the real numbers $\mathbb{R}$ with the dimension $\dim(V) = n < \infty$. Let $f : V \to V$ be a linear mapping. Assume that f has the property $f \circ f = -\mathrm{id}_V$, where id_V denotes the identity mapping of V.

a) Prove that f has no (real) eigenvalues.

b) Let $e \in V$, $e \neq 0$. Prove that e and $f(e)$ are linearly independent.

c) Prove that the dimension n of V must be even.

3. Consider the following matrices in $M_3(\mathbb{C})$.

$$\text{(i)} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{(ii)} \begin{pmatrix} 3 & 0 & 0 \\ 2 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{(iii)} \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

a) Find the characteristic polynomials $C(x)$ for each of the matrices above.

b) Find the minimal polynomials $M(x)$ for each of the matrices above.

c) Find the Jordan canonical form for each of the matrices above.

4. a) Describe the automorphism group $Aut(\mathbb{Z}/9\mathbb{Z})$ of the group $\mathbb{Z}/9\mathbb{Z}$.

b) Give an example of a group with the trivial center.

c) Prove that if a group G has the trivial center then $|Aut(G)| \geq |G|$.

5. Let S_n denote the symmetric group on n letters.

a) Find an element $a \in S_n$ of order n.

b) Find an element $b \in S_5$ of order 6.

c) Describe the conjugacy classes in S_4. How many are there?

6. Let G be the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by elements $(2,6), (8,21), (-4,-9)$. Find two generators for G.

7. a) Let R be a commutative ring, $I \subset R$ an ideal such that R/I is a field. Let $a, b \in R$ such that $ab \in I$. Prove that $a \in I$ or $b \in I$.

b) Describe all maximal ideals in the polynomial ring $\mathbb{C}[x]$.

c) Show that the principal ideal $(y) \subset \mathbb{C}[x,y]$ is prime, but not maximal.

d) Show that the principal ideal $(y - x^2) \subset \mathbb{C}[x,y]$ is prime.

8. Consider the ring $M_n(\mathbb{R})$ of $n \times n$ matrices with real entries.

a) Show that $M_n(\mathbb{R})$ is a simple ring (has no nontrivial two-sided ideals).

b) Let M be a left module over $M_n(\mathbb{R})$. Notice that the field $\mathbb{R}$ is embedded in the ring $M_n(\mathbb{R})$ as scalar matrices; hence in particular M is an $\mathbb{R}$-vector space. Prove that $dim_{\mathbb{R}}(M) \geq n$. (Hint: use a)).

9. a) Let $\mathbb{F}_q$ denote a finite field with q elements. Prove that $q = p^n$ for a prime p and some $n > 0$. (You cannot use the classification of finite fields.)

b) Describe the Galois group $Gal(\mathbb{F}_8/\mathbb{F}_2)$ explicitly.

c) Let $f(x) \in \mathbb{F}_2[x]$ be a polynomial of degree 5. Show that the field $\mathbb{F}_{32}$ contains all the roots of $f(x)$.