TIER ONE ALGEBRA EXAM

(1) Give an example of each of the following. (No justification required.)

(a) A group G, a normal subgroup H of G, and a normal subgroup K of H such that K is not normal in G.
(b) A non-trivial perfect group. (Recall that a group is perfect if it's only it has no non-trivial abelian quotient groups.)
(c) A 2×2 matrix A over $\mathbb{R}$ which is not diagonalizable over $\mathbb{R}$.
(d) A field which is a three dimensional vector space over the field of rational numbers, $\mathbb{Q}$.
(e) A group with the property that the subset of elements of finite order is not a subgroup.
(f) A prime ideal of $\mathbb{Z} \times \mathbb{Z}$ which is not maximal.

(2) A element a in a ring R is called *idempotent* if $a^2 = a$. Show the the only idempotent elements in an integral domain are 0 and 1.

(3) Consider the matrix $A = \begin{pmatrix} -4 & 18 \\ -3 & 11 \end{pmatrix}$.

(a) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
(b) Using the previous part of this problem, find a formula for A^n where A^n is the result of multiplying A by itself n times.
(c) Consider the sequences of numbers

$$a_0 = 1, b_0 = 0, a_{n+1} = -4a_n + 18b_n, b_{n+1} = -3a_n + 11b_n.$$

Use the previous parts of this problem to compute a closed formulae for the numbers a_n and b_n.

(4) Let

$$R = \frac{\mathbb{C}[x, y]}{(x^2 + y^3)}$$

where $\mathbb{C}[x, y]$ is the polynomial ring over the complex numbers $\mathbb{C}$ with indeterminates x and y. Similarly, let S be the subring of $\mathbb{C}[t]$ given by $\mathbb{C}[t^2, t^3]$.

(a) Prove that R and S are isomorphic as rings.
(b) Let I be the ideal in R given by the residue classes of x and y. Prove that I is a prime ideal of R but not a principle ideal of R.

Date: August 2003.

(5) Suppose that $T : R^n \to R^n$ is a linear map.

(a) Suppose $n = 2$ and $T^2 = -I$. Prove that T has no eigenvectors in R^2.
(b) Suppose $n = 2$ and $T^2 = I$. Prove that R^2 has a basis consisting of eigenvectors of T.
(c) Suppose $n = 3$. Prove that T has an eigenvector in R^3. (**I suggest we omit this last part to shorten the problem: Give an example of an operator T such that T has an eigenvector in R^3, but R^3 does not have a basis consiting of eigenvectors of T.**)

(6) Suppose that W is a non-zero finite dimensional vector space over $\mathbb{R}$. Let T be a linear transformation of W to itself. Prove that there is a subspace U of W of dimension 1 or 2 such that $T(U) \subset U$ (i.e. U is an invariant subspace. Here $T(U)$ denotes the set $\{T(u) | u \in U\}$.)

(7) Let $p(x)$ and $q(x)$ be polynomials with rational coefficients such that $p(x)$ is irreducible over the field of rational numbers $\mathbb{Q}$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be the complex roots of p, and suppose that $q(\alpha_1) = \alpha_2$. Prove that

$$q(\alpha_i) \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

for all $i \in \{2, 3, \ldots, n\}$.

(8) Let F be a field containing subfields F_{16} and F_{64} with 16 and 64 elements respectively. Find (with proof) the order of $F_{16} \cap F_{64}$.

(9) Let G be a finite group and suppose H is a subgroup of G having index n. Show there is a normal subgroup K of G with $K \subset H$ and such that the order of K divides $n!$.

(10) Let P_2 be the vector space of degree less than or equal to 2 polynomials with real coefficients. Define $D : P_2 \to P_2$ by $D(f) = f'$, that is, D is the linear transformation given by taking the derivative of the polynomial f. (You needn't verify that D is a linear transformation.)

Find a matrix representing the linear function D in the basis $\{1, x, x^2\}$. Determine the eigenvalues and eigenvectors of D. Determine if P_2 has a basis such that D us represented by a diagonal matrix. Why or why not?