TIER ONE ALGEBRA EXAM

(1) Consider the matrix $A = \begin{pmatrix} -4 & 18 \\ -3 & 11 \end{pmatrix}$.

(a) Find an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix.
(b) Using the previous part of this problem, find a formula for $A^n$ where $A^n$ is the result of multiplying $A$ by itself $n$ times.
(c) Consider the sequences of numbers
   
   $a_0 = 1, b_0 = 0, a_{n+1} = -4a_n + 18b_n, b_{n+1} = -3a_n + 11b_n.$
   
   Use the previous parts of this problem to compute closed formulae for the numbers $a_n$ and $b_n$.

(2) Let $P_2$ be the vector space of polynomials with real coefficients and having degree less than or equal to 2. Define $D : P_2 \rightarrow P_2$ by $D(f) = f'$, that is, $D$ is the linear transformation given by taking the derivative of the polynomial $f$. (You needn’t verify that $D$ is a linear transformation.)

(3) Give an example of each of the following. (No justification required.)

(a) A group $G$, a normal subgroup $H$ of $G$, and a normal subgroup $K$ of $H$ such that $K$ is not normal in $G$.
(b) A non-trivial perfect group. (Recall that a group is perfect if it has no non-trivial abelian quotient groups.)
(c) A field which is a three dimensional vector space over the field of rational numbers, $\mathbb{Q}$.
(d) A group with the property that the subset of elements of finite order is not a subgroup.
(e) A prime ideal of $\mathbb{Z} \times \mathbb{Z}$ which is not maximal.

(4) Show that any field with four elements is isomorphic to $\frac{\mathbb{F}_2[t]}{(1 + t + t^2)}$.

(5) Let $\mathbb{F}_p$ denote a field with $p$ elements, $p$ prime. Consider the ring

$R = \frac{\mathbb{F}_p[x, y]}{< x^2 - 3, y^2 - 5 >}$

where $< x^2 - 3, y^2 - 5 >$ denotes the ideal generated by $x^2 - 3$ and $y^2 - 5$. Show that $R$ is not a field.

(6) Let

$R = \frac{\mathbb{C}[x, y]}{(x^2 + y^3)}$

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where \( \mathbb{C}[x, y] \) is the polynomial ring over the complex numbers \( \mathbb{C} \) with indeterminates \( x \) and \( y \). Similarly, let \( S \) be the subring of \( \mathbb{C}[t] \) given by \( \mathbb{C}[t^2, t^3] \).

(a) Prove that \( R \) and \( S \) are isomorphic as rings.

(b) Let \( I \) be the ideal in \( R \) given by the residue classes of \( x \) and \( y \). Prove that \( I \) is a prime ideal of \( R \) but not a principle ideal of \( R \).

(7) Suppose that \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a linear map.

(a) Suppose \( n = 2 \) and \( T^2 = -I \). Prove that \( T \) has no eigenvectors in \( \mathbb{R}^2 \).

(b) Suppose \( n = 2 \) and \( T^2 = I \). Prove that \( \mathbb{R}^2 \) has a basis consisting of eigenvectors of \( T \).

(c) Suppose \( n = 3 \). Prove that \( T \) has an eigenvector in \( \mathbb{R}^3 \). Give an example of an operator \( T \) such that \( T \) has an eigenvector in \( \mathbb{R}^3 \), but \( \mathbb{R}^3 \) does not have a basis consisting of eigenvectors of \( T \).

(8) Let \( p(x) \) and \( q(x) \) be polynomials with rational coefficients such that \( p(x) \) is irreducible over the field of rational numbers \( \mathbb{Q} \). Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \) be the complex roots of \( p \), and suppose that \( q(\alpha_1) = \alpha_2 \). Prove that

\[
q(\alpha_i) \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}
\]

for all \( i \in \{2, 3, \ldots, n\} \).

(9) Let \( F \) be a field containing subfields \( F_{16} \) and \( F_{64} \) with 16 and 64 elements respectively. Find (with proof) the order of \( F_{16} \cap F_{64} \).

(10) Let \( G \) be a finite group and suppose \( H \) is a subgroup of \( G \) having index \( n \). Show there is a normal subgroup \( K \) of \( G \) with \( K \subset H \) and such that the order of \( K \) divides \( n! \).

(a) Find a matrix representing the linear function \( D \) in the basis \( \{1, x, x^2\} \).

(b) Determine the eigenvalues and eigenvectors of \( D \).

(c) Determine if \( P_2 \) has a basis such that \( D \) is represented by a diagonal matrix. Why or why not?

(11) Suppose that \( W \) is a non-zero finite dimensional vector space over \( \mathbb{R} \). Let \( T \) be a linear transformation of \( W \) to itself. Prove that there is a subspace \( U \) of \( W \) of dimension 1 or 2 such that \( T(U) \subset U \) (i.e. \( U \) is an invariant subspace. Here \( T(U) \) denotes the set \( \{T(u) | u \in U\} \).)