Tier 1 Examination - Algebra
January 4, 2001

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \( \mathbb{R} \). The set of integers modulo \( n \) is denoted \( \mathbb{Z}_n \). The group of permutations on \( n \) letters is denoted \( S_n \).

(20)1. Give an example of each of the following. No justification is required.
(a) A nonabelian group of order 18.
(b) An infinite commutative ring \( R \) such that for all \( y \in R \), \( y + y + y = 0 \).
(c) A 3 by 3 real matrix that is diagonalizable over the complex numbers but not over the reals.
(d) A unique factorization domain that is not a principal ideal domain.
(e) An element of order 3 in \( \text{GL}_2(\mathbb{R}) \).

(10) 2. Let \( G \) be a group with the property that \( g^2 = e \) for all \( g \in G \). Prove \( G \) is abelian.

(10)3. Determine the number of homomorphisms from \( S_3 \) to \( \mathbb{Z}_2 \times \mathbb{Z}_4 \).

(10)4. Find \( \lim_{n \to \infty} \left( \begin{array}{cc} 2 & 3 \\ -1/2 & -1/2 \end{array} \right)^n \).

(10)5. Let \( n \) be a positive odd integer and let \( A, B \in \mathcal{M}_n(\mathbb{R}) \) such that \( A^2 = B^2 = I \). Prove that \( A \) and \( B \) have a common eigenvector (not necessarily with the same eigenvalue).

(10)6. Let \( R \) and \( S \) be commutative rings and let \( \phi : R \to S \) be a ring homomorphism. Suppose there is an ideal \( I \) of \( R \) such that \( \ker(\phi) \subset I \subset R \) (proper containments). Prove that the image of \( \phi \) is not a field.

(10)7. Let \( F \) be a subfield of \( \mathbb{R} \) and suppose \( m \) and \( n \) are positive integers with \( \sqrt{m} + \sqrt{n} \in F \). Prove that \( \sqrt{m} \) and \( \sqrt{n} \) are in \( F \).

(10)8. Let \( K \) be a field and let \( K^\times \) denote the group of nonzero elements of \( K \). Prove that \( K^\times \) contains at most two elements of order 6.
(10)9. Let $G = (\mathbb{Q}, +)/(\mathbb{Z}, +)$, where $(\mathbb{Q}, +)$ denotes the group of rational numbers under addition and $(\mathbb{Z}, +)$ denotes the subgroup of integers under addition. Prove that $G$ is an infinite group in which every element has finite order.

(15)10. (a) Let $R$ be a commutative ring and suppose $I$ and $J$ are ideals of $R$ such that $I + J = R$, where $I + J = \{i + j | i \in I, j \in J\}$. Prove the map $\phi: R/I \cap J \to R/I \times R/J$ given by $\phi(r + I \cap J) = (r + I, r + J)$ is an isomorphism of rings.

(b) Let $R$ be a commutative ring containing exactly 4 ideals (including $\{0\}$ and $R$). Let $I$ and $J$ denote the other two ideals and suppose they are incomparable, that is $I \not\subseteq J$ and $J \not\subseteq I$. Prove that $R$ is isomorphic to the direct product of two fields.