Tier One Algebra Exam

January 1999

4 Hours

Each problem is 10 points.

1. Find a invertible matrix M such that $M^{-1}AM$ is diagonal, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Give an example of a 2×2 matrix that does not have two linearly independent eigenvectors.

3. A square matrix A is called nilpotent if $A^k = 0$ for some $k > 0$.

   a) Give an example with justification of a nonzero nilpotent A.

   b) Prove that if A is nilpotent, then $I + A$ is invertible.

4. Let G be a finite group, $a, b \in G$. Prove that the orders of ab and ba are equal.

5. Prove that the set of elements of finite order in an abelian group is a subgroup.

6. Let G be a finite group of order > 2. Prove that G has a nontrivial automorphism.

7. Find two generators for the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $\{(8,7),(2,5),(9,3)\}$.

8. Consider the ring homomorphism

$$f : \mathbb{Z}[x] \to \mathbb{R}$$

   which maps x to $\sqrt[3]{2}$. Consider the ideal $Ker(f)$ in $\mathbb{Z}[x]$. Show $Ker(f)$ is generated by a single polynomial and find that polynomial.

9. Prove that the ring $H := \mathbb{F}_2[x]/(x^3 + x^2 + 1)$ is a field. Find the degree of the field extension $[H : \mathbb{F}_2]$.

10. Let R be a commutative ring and $I \subset R$ be an ideal. Consider the set

$$J := \{x \in R | x^n \in I \text{ for some } n \geq 1\}.$$

a) Show that J is an ideal in R.

An ideal I is called **primary** if for all x and y satisfying $xy \in I$, either $x \in I$ or $y^m \in I$ for some $m \geq 1$, where m may depend on y.

b) Show that if I is primary, J is prime.

11. Show that if some element of a commutative ring has three or more square roots, the ring is not an integral domain.