Tier 1 Algebra Examination

August, 1997.

Time: 3 Hours

Question 1 is worth 25 points, questions 2 through 4 are worth 10 points each and questions 5 through 7 are worth 15 points each.

Start each question on a fresh sheet of paper.

1. Give examples (no need to prove anything) or give mathematical reasons if you can not give examples:

   a. An infinite group all of whose elements have orders 1 or 3.

   b. Matrices A and B of sizes 3 x 2 and 2 x 3 respectively such that $A \cdot B = I_3$, (identity matrix of size 3 x 3)

   c. Two elements in the alternating group A_5 which are conjugate in the symmetric group S_5 but not in A_5.

   d. An u.f.d. which is not a p.i.d.

   e. A transcendental element $\alpha \in \mathbb{C}$ such that $\alpha - \frac{1}{\alpha}$ is an algebraic element.

2. a. Let f be an automorphism of the group $\mathbb{Z}/16\mathbb{Z}$. Show that there exists an odd integer m ($1 \le m \le 15$) such that $f(x) = mx$ for all $x \in \mathbb{Z}/16\mathbb{Z}$.

   b. Decompose the group $\mathrm{Aut}(\mathbb{Z}/16\mathbb{Z})$ of all automorphisms of $\mathbb{Z}/16\mathbb{Z}$ as a product of cyclic groups.

3. a. Let G be a group and H be a subgroup. Let $N = \{ g \in G \mid gHg^{-1} = H \}$ be the normalizer of H in G. Show that there is a bijective correspondence between the left cosets of N in G and the set $\mathscr{S} = \{ K \mid K = xHx^{-1} \text{ for some } x \in G \}$ of all conjugates of H.

   b. Let G, H, and N be as in (a) above. If in addition, G is finite with $|H| = r$ and $[G:H] = s$, then show that the union of all members of $\mathscr{S}$ (i.e. $\bigcup_{K \in \mathscr{S}} K$) has at most $(rs - s + 1)$ elements.

4. Let V be an n–dimensional vector space.

a. Show that a proper subspace W of V is the intersection of all subspaces of V of dimension n–1 which contain W.

b. Let A(V) be the vector space of all linear transformations of V to itself. For $x \neq 0$ in V, compute the dimension of $A_x(V) = \{ T \in A(V) \mid T(x) = 0 \}$.

5. Let R be a commutative ring with 1. For an ideal I of R, define $\sqrt{I}$ to be the set $\{ x \in R \mid x^n \in I \text{ for some integer } n \geq 1 \}$.

a. Show that $\sqrt{I}$ is an ideal of R which contains I.

b. If I is a prime ideal, show that $\sqrt{I} = I$.

c. If R is a u.f.d and x is a non–zero, non–unit element in R, find a y such that $\sqrt{R.x} = R.y$. (Hint: consider a prime power factorization for x).

6. Let R be an Euclidean domain with a valuation v (i.e. v is a function from the set of non–zero elements of R to the set of non–negative integers such that (i) $v(x) \leq v(xy)$ for $x, y \in R \setminus \{0\}$ and (ii) given $z \in R$ and $y \in R \setminus \{0\}$, there exist q and r such that $z = yq + r$ with $r = 0$ or $v(r) < v(y)$). Assume further that $v^{-1}(n)$ is finite for all n.

a. Show that for any non–zero ideal I, R/I is finite. (Note that I = R.y for some y).

b. For the ring $\mathbb{Z}[i] = \{ a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z} \}$ of Gaussian integers with standard valuation v given by $v(a + bi) = a^2 + b^2$, prove that $\mathbb{Z}[i] \,/\, 3\cdot\mathbb{Z}[i]$ is a field of 9 elements. (Hint: show that 3 is a prime)).

7. Let p be a prime and n be a positive integer relatively prime to p. Let K be the splitting field of $x^n - 1$ over F_p, the prime field of p elements. Let $[K : F_p] = m$.

a. Show that n divides $p^m - 1$. (a hint is given below)

b. If r is such that n divides $p^r - 1$, show that $m \leq r$.

(Hint for parts (a) and (b): Show first that roots of $x^n - 1$ are all distinct and they form a subgroup of the multiplicative group $K \setminus \{0\}$ which is cyclic).

c. Find $[K : F_3]$ where K is the splitting field of $x^{14} - 1$.