Geometric Properties of Conformal Transformations on $\mathbb{R}^{p,q}$

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1 Description:

Let $p, q = 0, 1, 2, \dots$, and let $\mathbb{R}^{p,q}$ denote the Euclidean vector space $\mathbb{R}^{p+q}$ with indefinite quadratic form

$$Q_{p,q}(\overrightarrow{v}) = (x_1)^2 + \dots + (x_p)^2 - (x_{p+1})^2 - \dots - (x_{p+q})^2.$$

We denote by $O(p,q)$ the set of $(p+q) \times (p+q)$ real matrices preserving this quadratic form, i.e. $O(p,q)$ consists of all real matrices A such that $Q_{p,q}(A\overrightarrow{v}) = Q_{p,q}(\overrightarrow{v})$.

For each $(p+q+2) \times (p+q+2)$ matrix in $O(p+1, q+1)$, there is a way to produce a transformation on $\mathbb{R}^{p,q}$ which is *conformal*. (Conformal transformations generalize metric-preserving transformations in differential geometry.)

Here is an example typically covered in a complex variables course (such a course is *not* a prerequisite). Let $p = 2$, $q = 0$ and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. For each invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{C}$, one can define two *fractional linear transformations* sending

$$z \mapsto \frac{az+b}{cz+d} \quad \text{and} \quad z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$$

($\bar{z}$ denotes the complex conjugate of z), and each of these transformations is conformal on $\mathbb{C} \simeq \mathbb{R}^2$. These are exactly the same transformations as those produced by $O(3,1)$. The fractional linear transformations have the following important geometric property – they map circles and lines in $\mathbb{C}$ into circles and lines.

The goal of this project is to investigate geometric properties of conformal transformations on $\mathbb{R}^{p,q}$ and, in particular, see if this "mapping circles to circles" property generalizes to other values of p and q. While initially this problem involved differential geometry, it is now reduced to linear algebra and multivariable calculus. This project can possibly produce new mathematical results publishable in a research journal.

2 Prerequisites:

Strong background in linear algebra (including quadratic forms) and multivariable calculus are absolutely necessary. Exposure to a first course in differential geometry would help to understand the motivation behind the project, but is not a prerequisite.