Research Experience for Undergraduates Program
Research Reports

Indiana University, Bloomington

Summer 2018

Contents

Preface

During the summer of 2018 eight students participated in the Research Experience for Undergraduates program in Mathematics at Indiana University. This program was sponsored by the National Science Foundation through the Research Experience for Undergraduates grant #1757857 and the Department of Mathematics at Indiana University, Bloomington. The program ran for eight weeks, from June 4 through July 27, 2018. Eight faculty served as research advisers to the students from Indiana University:

- Eric Albers was advised by Nicholas Miller.
- Emi Brawley was advised by Noah Snyder.
- Cornell Holmes was advised by Chris Connell.
- Matisse Peppet was advised by Alex Kruckman.
- Jacob Prinz was advised by Nachiket Karnick and Noah Snyder.
- James Reber was advised by Graham White.
- Ben Riley was advised by Carmen Rovi.
- Fan Zhou was advised by Chris Connell.

Following the introductory pizza party, students began meeting with their faculty mentors and continued to do so throughout the next eight weeks. The students also participated in a number of social events and educational opportunities and field trips.

Individual faculty gave talks throughout the program on their research, about two a week. Students also received LaTeX training in a series of workshops. Other opportunities included the option to participate in a GRE and subject test preparation seminar. Additional educational activities included tours of the library, the Slocum puzzle collection at the Lilly Library, and self guided tours of the art museum. Students presented their work to faculty mentors and their peers at various times. This culminated in their presentations both in poster form and in talks at the statewide Indiana Summer Undergraduate Research conference which we hosted at the Bloomington campus of IU.

On the lighter side, students were treated to weekly board game nights as well as the opportunity to do some local hiking. They also enjoyed a night of "laser tag" courtesy of the Department of Mathematics.

The summer REU program required the help and support of many different groups and individuals to make it a success. We foremost thank the National Science Foundation and the Indiana University Bloomington Department of Mathematics without whose support this program could not exist. We especially thank our staff member Mandie McCarty for coordinating the complex logistical arrangments (housing, paychecks, information packets, meal plans, frequent shopping for snacks). Additional logistical support was provided by the Department of Mathematics and our chair, Elizabeth Housworth. We are in particular thankful to Jeff Taylor for the computer support he provided. Thanks also go to those faculty who served as mentors and those who gave lectures. We thank David Baxter of the Center for Exploration of Energy and Matter (nee IU cyclotron facility) for past personal tours of the LENS facility and his informative lectures. Thanks to Andrew Rhoda for his tour of the Slocum Puzzle Collection.

Chris Connell
September, 2018

Figure 1: REU Participants, from left to right: Cornell Holmes, James Reber, Jacob Prinz, Emi Brawley, Chris Connell, Ben Riley, Matisse Peppet, Eric Albers, Fan Zhou.

GENERA OF CONGRUENCE SURFACES

ERIC ALBERS

ABSTRACT. The purpose of this paper is to show the impossibility of a genus 212 congruence surface constructed from a maximal order in a quaternion algebra using bounds on the values for $\zeta_K(-1)$ and Sage to computationally check all possible number fields up to our bounds. From this data, we obtain a list of all possible number fields in which the zeta function allows for the possibility of a genus 212 surface and provide an argument for these remaining fields as to why no such construction exists.

1. INTRODUCTION

Denote the Teichmüller space of genus g with n cusps by $T_{g,n}$. An open question in geometry is to find the manifolds $M \in T_{g,n}$ which maximize the length of the systole for arbitrary g, n. Schmutz showed in [8] that in the noncompact setting, congruence surfaces maximize the systole in their Teichmüller spaces. In the compact setting, it is not unreasonable to expect a similar result can be obtained, as Katz, Schaps, and Vishne provided a particularly sharp lower bound on the length of the systole of a compact congruence surface in [3]. Our knowledge of congruence surfaces, however, is still incomplete. While Sebbar classified all genus 0 congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$ in [9], it is still not known whether congruence surfaces exist in every genus. As a result, Reid asked the following.

Question 1.1. *Let M be a congruence surface. What is the genus of M, in particular if we denote by S the set of all such congruence surfaces, is it true that*

$$T = \{gen(M) | M \in S\}$$

consists of all natural numbers ≥ 2.

In order to answer this question, one must first understand the genus of an arbitrary congruence surface which is constructed from a maximal order in a quaternion algebra. Thus, in an attempt to understand Question 1.1, this paper analyzes the following related question.

Question 1.2. *Let M be a congruence surface constructed from a maximal order in a quaternion algebra. If S is the set of all such surfaces, is it true that*

$$T = \{gen(M) | M \in S\}$$

consists of all natural numbers ≥ 2.

The purpose of this paper is to answer Question 1.2 negatively with a proof of the following theorem.

Theorem 1.3. *There is no congruence surface constructed from a maximal order of genus 212.*

Acknowledgements The author would like to thank Indiana University - Bloomington for their hospitality and accommodations during this REU, specifically Professor Chris Connell for organizing a fantastic program. They would also like to thank the National Science Foundation for their grant which funded this project. Finally they would like to thank Professor Miller for his mentorship and general guidance throughout the entirety of the project.

2. Background Material

2.1. **Arithmetic.** A number field K is a finite extension of $\mathbb{Q}$, its degree we will denote by $[K:\mathbb{Q}]$, and K is totally real if for any embedding $\sigma : K \hookrightarrow \mathbb{C}$ (of which there are $[K:\mathbb{Q}]$), $\sigma(K) \subset \mathbb{R}$. The ring of algebraic integers $\mathcal{O}_K \subset K$ is the ring consisting of elements of K which are the root of a monic polynomial with integer coefficients. It is well known that for any number field K, its integer ring $\mathcal{O}_K$ is a free abelian group of rank $[K:\mathbb{Q}]$ under addition, giving that it has a basis $\{\beta_1, \dots, \beta_n\}$ over $\mathbb{Z}$, and such a basis is also a basis for K over $\mathbb{Q}$. We say $\{\beta_1, \dots, \beta_n\}$ is an integral basis. Let K be a number field of degree n, $\{\beta_1, \dots, \beta_n\}$ an integral basis, and $\sigma_1, \dots, \sigma_n$ be the n embeddings $K \to \mathbb{C}$. The discriminant for $\{\beta_1, \dots, \beta_n\}$ is given by

$$\operatorname{disc}(\beta_1, \dots, \beta_n) = |\sigma_i(\beta_j)|^2$$

and a simple matrix argument shows that for a given number field K, the discriminant of any two integral bases is equal. We denote by Δ_K the discriminant for any integral basis of K.

$\mathcal{O}_K$ is also a Dedekind domain giving that ideals in $\mathcal{O}_K$ factor uniquely into a product of prime ideals. Let L/K be an extension of number fields $\mathfrak{p} \subset \mathcal{O}_K$ some prime ideal and denote by $\mathfrak{p}\mathcal{O}_L$ the ideal in $\mathcal{O}_L$ generated by elements of $\mathfrak{p}$. Unique factorization tells us that

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \dots \mathfrak{q}_n^{e_n}$$

We say $\mathfrak{q}_i$ lies over $\mathfrak{p}$ or $\mathfrak{p}$ lies under $\mathfrak{q}_i$, often notated $\mathfrak{q}_i|\mathfrak{p}$, for $i = 1, \dots, n$ and the exponent $e_i = e(\mathfrak{q}_i|\mathfrak{p})$ is known as the ramification index for $\mathfrak{q}_i$ over $\mathfrak{p}$. $\mathfrak{p}$ is said to be ramified in L, or ramifies in L, if there exists a q_i lying over $\mathfrak{p}$ with $e(\mathfrak{q}_i|\mathfrak{p}) > 1$. It is a standard result that any prime $\mathfrak{q} \subset \mathcal{O}_L$ lies over a unique prime $\mathfrak{p} \subset \mathcal{O}_K$, and that the integral primes that ramify in a number field K, are precisely those $p \in \mathbb{Z}$ such that $p|\Delta_K$. We also have a natural injection $\mathcal{O}_K/\mathfrak{p} \hookrightarrow \mathcal{O}_L/\mathfrak{q}_i$ and since quotients of Dedekind domains by prime ideals are finite fields, we see that $\mathcal{O}_L/\mathfrak{q}_i$ is a finite extension of $\mathcal{O}_K/\mathfrak{p}$. The degree of this extension $[\mathcal{O}_L/\mathfrak{q}_i : \mathcal{O}_K/\mathfrak{p}] = f = f(\mathfrak{q}_i|\mathfrak{p})$ is called the inertial degree of $\mathfrak{q}_i$ over $\mathfrak{p}$. A fundamental result of number theory is the following

Theorem 2.1. *Let L/K be an extension of number fields, $p \subset \mathcal{O}_K$ some prime, $\mathfrak{q}_1, \dots, \mathfrak{q}_n \subset \mathcal{O}_L$ the distinct primes lying over $\mathfrak{p}$. Then*

$$[L:K] = \sum_{i=1}^{n} e_i f_i$$

The norm of an ideal $I \subset \mathcal{O}_K$ is given by $N(I) = |\mathcal{O}_K/I|$. Since prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ lie over unique primes in $\mathbb{Z}$,$\mathcal{O}_K/\mathfrak{p}$ is a finite extension of $\mathbb{Z}/p\mathbb{Z}$ and hence $N(\mathfrak{p}) = p^f$ where f is the inertial degree $f(\mathfrak{p}|p)$.

A place v on a number field K is an equivalence class of valuations on K, and it is a theorem of Onstrowski that the unique places on K are given by $v_\sigma(x) = |\sigma(x)|$ for any of the $[K:\mathbb{Q}]$ distinct embeddings $K \hookrightarrow \mathbb{C}$ and the $\mathfrak{p}$-adic valuations where $\mathfrak{p} \subset \mathcal{O}_K$ is a prime ideal. The places given by embeddings are said to be infinite and the $\mathfrak{p}$-adic places finite. For any infinite place v, the completion $K_v \cong \mathbb{R}$ or $\mathbb{C}$ depending on whether $\sigma(K) \subset \mathbb{R}$ and we will denote by $K_\mathfrak{p}$ the $\mathfrak{p}$-adic completion of K.

Definition 2.2. *The ring of integers, $\mathcal{O}_\mathfrak{p}$, of $K_\mathfrak{p}$ is the valuation ring*

$$\{x \in K_\mathfrak{p} | \nu(x) \geq 1\}$$

where ν is the logarithmic $\mathfrak{p}$-adic valuation.

Let K be a number field, $\mathfrak{p} \subset \mathcal{O}_K$ be prime, $K_\mathfrak{p}$ the corresponding completion of K. The ring of integers, $\mathcal{O}_\mathfrak{p}$ of $K_\mathfrak{p}$ has a unique maximal ideal $I = \{x \in \mathcal{O}_\mathfrak{p} | \nu(x) > 1\}$ which is generated by a single element, often called the uniformiser, by which we will denote π.

Definition 2.3. *Let K be a number field. The Dedekind zeta function for K is given by*

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s}$$

where I runs through all non-zero ideals in the ring of integers $\mathcal{O}_K, s \in \mathbb{C}$.

Like the Riemann zeta function, there is a product expression in terms of prime ideals, as ideals factor uniquely into primes in integer rings, given by

$$\zeta_K(s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - \frac{1}{N(\mathfrak{p})^s}}$$

where $\mathfrak{p}$ runs through all prime ideals in $\mathcal{O}_K$.

2.2. **Quaternion Algebras over Number Fields.**

Definition 2.4. *An F-quaternion algebra $\mathscr{A}$ is a 4-dimensional vector space over F with basis $\{1, i, j, ij\}$ where multiplication is extended linearly such that 1 is the multiplicative identity and*

$$i^2 = a \quad j^2 = b \quad ij = -ji$$

for some $a, b \in F^*$

Quaternion algebras are central and simple and are the only 4-dimensional central simple algebras. We will use the notation $\left(\frac{a,b}{F}\right)$ to denote the F-quaternion algebra where $i^2 = a, j^2 = b$.

Definition 2.5. *Given $\alpha = x_0 + x_1 i + x_2 j + x_3 ij \in \mathscr{A}$, its conjugate is given by $\overline{\alpha} = x_0 - x_1 i - x_2 j - x_3 ij$. The norm of $\alpha \in \mathscr{A}$ is given by $n(\alpha) = \alpha\overline{\alpha}$ and the trace by $tr(\alpha) = \alpha + \overline{\alpha}$.*

The norm 1 elements, denoted $\mathscr{A}^1$, of a quaternion algebra form a group under multiplication. There is a corresponding notion of integral elements in quaternion algebras over number fields. $\alpha \in \mathscr{A}$ is an integer if $\mathcal{O}_K[\alpha]$ is finitely generated, which is equivalent to requiring both $n(\alpha), tr(\alpha) \in \mathcal{O}_K$. We have the following classification of quaternion algebras over $\mathbb{R}$

Theorem 2.6. *Let $\mathscr{A} = \left(\frac{a,b}{\mathbb{R}}\right)$ be a quaternion algebra over $\mathbb{R}$. Then*

$$\mathscr{A} \cong M_2(\mathbb{R}) \text{ or } \mathcal{H}$$

where $\mathscr{A} \cong M_2(\mathbb{R})$ if either one of $a, b > 0$ and otherwise $\mathscr{A} \cong \mathcal{H}$, where $\mathcal{H}$ denotes Hamilton's quaternions.

Let K be a totally real number field, $\mathscr{A} = \left(\frac{a,b}{K}\right)$, σ be an embedding into $\mathbb{R}$, and v the place corresponding to σ, then

$$\mathscr{A} \otimes_K K_v \cong \left(\frac{\phi(a), \phi(b)}{\mathbb{R}}\right)$$

where by ϕ we mean the map $\phi : K \to \mathbb{R}$ which is the composition of the inclusion map into K_v and the isomorphism between K_v and $\mathbb{R}$. $\mathscr{A}$ is said to be split at v if $\mathscr{A} \otimes_K K_v \cong M_2(\mathbb{R})$ and ramified if $\mathscr{A} \otimes_K K_v \cong \mathcal{H}$. For $\mathfrak{p}$-adic completions we have a similar analogue.

Theorem 2.7. *Let K be a number field, $\mathfrak{p} \subset \mathcal{O}_K$ be prime, $\mathscr{A}$ be a $K_{\mathfrak{p}}$-quaternion algebra. Then*

$$\mathscr{A} \cong M_2(K_{\mathfrak{p}}) \text{ or } \left(\frac{\pi, u}{K_{\mathfrak{p}}}\right)$$

where $K_{\mathfrak{p}}(\sqrt{u})$ is the unique unramified quadratic extension of $K_{\mathfrak{p}}$.

By unramified we mean the unique prime ideal $I \subset \mathcal{O}_{\mathfrak{p}}$ does not ramify in the extension. Likewise $\mathscr{A}$ is said to split at the place corresponding to $\mathfrak{p}$ if $\mathscr{A} \otimes_K K_{\mathfrak{p}} \cong M_2(K_{\mathfrak{p}})$ and ramifies otherwise. The set of places at which $\mathscr{A}$ ramifies, denoted $\mathrm{Ram}(\mathscr{A})$, is always finite and of even cardinality and, in fact, all quaternion algebras over a number field K can be classified based on their local behavior via the following theorem.

Theorem 2.8. *Let $\mathscr{A}, \mathscr{A}'$ be quaternion algebras over a number field K. $\mathscr{A} \cong \mathscr{A}'$ if and only if $Ram(\mathscr{A}) = Ram(\mathscr{A}')$. Moreover, there is a bijection between quaternion algebras over K and finite, even cardinality, subsets of the set of places on K.*

Definition 2.9. *Let $\mathscr{A}$ be a quaternion algebra over a number field K. An ideal $I \subset \mathscr{A}$ is a complete $\mathcal{O}_K$-lattice. An order $\mathcal{O}$ is an ideal that is a ring with 1.*

3. Hyperbolic Surfaces via Quaternion Algebras

In this section, we detail the way a hyperbolic surface is constructed from a quaternion algebra, define the congruence condition and provide the covolume formula that will be the focus of the rest of the paper. To begin we need the following theorem from Maclachlan–Reid [5, Theorem 8.1.1]

Theorem 3.1. *Let $\mathscr{A}$ be a quaternion algebra over a totally real number field K, of degree n, that is ramified at all but one of its real places. Then*

$$\mathscr{A} \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \oplus (n-1)\mathcal{H}$$

where $\mathcal{H}$ denotes Hamilton's quaternions.

Using this theorem we can now construct lattices in $\mathrm{PSL}_2(\mathbb{R})$ as follows. Let $\mathscr{A}$ be a quaternion algebra over a totally real number field K, of degree n, that is ramified at all but one of its real places. We may assume the embedding at which $\mathscr{A}$ splits is the identity, as otherwise the resultant lattice would differ only by a conjugate of the lattice formed via the identity. Let $\rho : \mathscr{A} \to M_2(\mathbb{R})$ be given by the composition of the inclusion map $\mathscr{A} \to \mathscr{A} \otimes_{\mathbb{Q}} \mathbb{R}$ and the projection onto the first term of the direct sum above. Given an order $\mathcal{O} \subset \mathscr{A}, \rho|_{\mathcal{O}^1} : \mathcal{O}^1 \hookrightarrow M_2(\mathbb{R})$ is an injective group homomorphism. Since $\mathscr{A}$ splits at the identity, if $\mathscr{A} = \left(\frac{a,b}{K}\right)$, we must have at least one of $a, b > 0$. Assuming $a > 0$ we can define ρ explicitly by the assignment

$$\alpha = x_0 + x_1 i + x_2 j + x_3 ij \mapsto \begin{pmatrix} x_0 + x_1\sqrt{a} & b(x_2 - x_3\sqrt{a}) \\ (x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix}$$

where we note $n(\alpha) = \det(\rho(\alpha))$ whether or not $b > 0$. Thus we actually have $\mathcal{O}^1 \hookrightarrow SL_2(\mathbb{R})$. Let $P : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ be the canonical quotient map. Then via the construction above, for any order $\mathcal{O} \subset \mathscr{A}, P(\rho(\mathcal{O}^1)) \leq \mathrm{PSL}_2(\mathbb{R})$. Moreover $P(\rho(\mathcal{O}^1))$ is discrete and of finite covolume, and provided $\mathscr{A}$ is a division algebra, $P(\rho(\mathcal{O}^1))$ is also cocompact [5, Theorem 8.1.2]. We remark here our algebra certainly must split at at least one infinite place in order to have such an embedding into $M_2(\mathbb{R})$, but if our algebra split at more than one place, the image of $\mathcal{O}^1$ in any copy of $M_2(\mathbb{R})$ would be dense and thus not discrete [5, Theorem 8.1.2]. Thus our assumption that K must split at only 1 infinite place is not only sufficient but also necessary. For the remainder of the paper, when we say a congruence surface is *constructed from a maximal order* we mean it is the surface given by the quotient of $\mathbb{H}^2$ by $P(\rho(\mathcal{O}^1))$ where P and ρ are as above and $\mathcal{O}^1$ is the group of norm 1 elements of a maximal order $\mathcal{O}$.

Definition 3.2. *An arithmetic Fuchsian group is a subgroup $\Gamma \subset PSL_2(\mathbb{R})$ that is commensurable with some $P(\rho(\mathcal{O}^1))$ where $\mathcal{O} \subset \mathscr{A}$ is an order in a quaternion algebra over a totally real number field K that ramifies at all but one real place and $P(\rho(\mathcal{O}^1))$ is as described above.*

Where Γ_1, Γ_2 are commensurable if there exists a space N that serves as a finite covering space for both $\mathbb{H}^2/\Gamma_1$ and $\mathbb{H}^2/\Gamma_2$. To simplify and slightly abuse notation, in the future when we write $\mathbb{H}^2/\mathcal{O}^1$, we implicitly mean $\mathbb{H}^2/(P(\rho(\mathcal{O}^1))$.

Definition 3.3. *Let $\mathscr{A}$ be a quaternion algebra, $\mathcal{O} \subset \mathscr{A}$ some maximal order, $I \subset \mathcal{O}$ an integral two-sided, ideal. The principal congruence subgroup of level I in $\mathcal{O}^1$ is given by*

$$\mathcal{O}^1(I) = \{x \in \mathcal{O}^1 | x - 1 \in I\}$$

That is to say $\mathcal{O}^1$ is the kernel of the homomorphism $\mathcal{O}^1 \to (\mathcal{O}/I)^*$.

An arithmetic Fuchsian group Γ is a congruence subgroup if it contains a principal congruence subgroup of some level. For a maximal order, $\mathcal{O} \subset \mathscr{A}$, $P(\rho(\mathcal{O}^1))$ is necessarily a congruence subgroup, as, by definition, it contains all principal congruence subgroups of $\mathcal{O}^1$. However, in general, not all arithmetic Fuchsian groups are congruence subgroups. To conclude this section, we provide the formula for the covolume of $\mathbb{H}^2/P(\rho(\mathcal{O}^1))$ where $\mathcal{O} \subset \mathscr{A}$ is a maximal order, from Borel [1]

Theorem 3.4. *Let K be a totally real number field, $\mathscr{A}$ a K-quaternion algebra that is ramified at all but one of its infinite places, $\mathcal{O} \subset \mathscr{A}$ some maximal order. Then the covolume of $\mathbb{H}^2/\mathcal{O}^1$ is given by*

$$\frac{8\pi\zeta_K(2)}{(4\pi^2)^{[K:\mathbb{Q}]}} \prod_{\mathfrak{p} \in Ram_f(\mathscr{A})} (N(\mathfrak{p}) - 1) \tag{1}$$

where $Ram_f(\mathscr{A})$ denotes the set of finite primes at which $\mathscr{A}$ ramifies.

The Gauss–Bonnet Theorem tells us, provided our surface is indeed a surface, this quantity should be an integer multiple of 4π. This leads to a necessary discussion on torsion in quaternion algebras. Elements of torsion in $\mathcal{O}^1$ lead to metric singularities in $\mathbb{H}^2/\mathcal{O}^1$, i.e. points where a smooth hyperbolic metric cannot be defined. We have the following relevant theorem from Maclachlan–Reid [5, Theorem 12.5.4] classifying when torsion occurs in the quaternion algebras we are interested in.

Theorem 3.5. *Let $\mathscr{A}$ be a quaternion division algebra over a number field K. The group $P(\rho(\mathscr{A}^1))$ contains an element of order n*

$\Leftrightarrow$ $\xi_{2n} + \xi_{2n}^{-1} \in K, \xi_{2n} \notin K$, and $L = K(\xi_{2n})$ embeds in $\mathscr{A}$.

$\Leftrightarrow$ $\xi_{2n} + \xi_{2n}^{-1} \in K, \xi_{2n} \notin K$ and if $\mathfrak{p} \in Ram_f(\mathscr{A})$ then $\mathfrak{p}$ does not split in $K(\xi_{2n})$.

where ξ_{2n} denotes the $2n^{th}$ root of unity. Moreover, if $P(\rho(\mathscr{A}^1))$ contains an element of n-torsion, then for all maximal orders $\mathcal{O} \subset \mathscr{A}$, $P(\rho(\mathcal{O}^1))$ contains an element of n-torsion.

By a simple degree argument, one easily sees that it must hold that $[L:K] \leq 2$ in order for L to embed into K, and since our fields are totally real, $[L:K] \neq 1$. For any totally real number field $K(\xi_4) = K(i)/K$ and $K(\xi_6) = K(\sqrt{-3})/K$ always have degree 2 over K and as a result we must always verify our algebra has no $2/3-$torsion. The second part of Theorem 3.5 tells us that the only way to avoid elements of torsion is to ramify at some prime $\mathfrak{p}$ which splits in $K(\xi_{2n})$, so from here forward when we speak of eliminating torsion, we precisely mean ramifying at such a prime $\mathfrak{p}$.

4. Congruence Conditions on Torsion in Quaternion Algebras

We begin this section by working through the simplest concrete case, where $K = \mathbb{Q}$, in order to understand the relation between torsion in $\mathscr{A}$ and the formula for the area of the resultant surface. $\Delta_K = [K : \mathbb{Q}] = 1, \zeta(2) = \frac{\pi^2}{6}$, and the prime ideals in $\mathbb{Z}$ are just the ideals generated by the rational primes, hence $N(p) = p$. Thus, for an arithmetic surface constructed from a maximal order in a quaternion algebra over $\mathbb{Q}$ we get its area to be

$$\frac{8\pi^3}{24\pi^2} \prod_{p \in \mathrm{Ram}_f(\mathscr{A})} (p-1)$$

and the Gauss–Bonnet Theorem tells us this quantity is $4\pi(g-1)$, giving

$$g - 1 = \frac{1}{12} \prod_{p \in \mathrm{Ram}_f(\mathscr{A})} (p-1) \tag{2}$$

We present now a lemma which will be useful in the case of general K, and will also help explain the connection between torsion and the denominators in our area formula.

Lemma 4.1. *For a totally real number field K, $\mathfrak{p} \subset K$ a non-dyadic prime ideal, if $\mathfrak{p}$ splits in $K(i)/K$, then $N(\mathfrak{p}) \equiv 1 \pmod 4$.*

Proof. This is trivially true if $\mathfrak{p}$ lies over a prime $p \equiv 1 \pmod 4$. If $\mathfrak{p}$ lies over some prime $p \equiv 3 \pmod 4$, $N(\mathfrak{p}) \equiv 1 \pmod 4$ if and only if $f(\mathfrak{p}|p) \equiv 0 \pmod 2$. Thus, suppose, by way of contradiction, $\mathfrak{p} \subset K$ lies over a prime $p \equiv 3 \pmod 4$, $f(\mathfrak{p}|p) \equiv 1 \pmod 2$, and $\mathfrak{p}$ splits in $K(i)/K$. Since $K(i)/K$ is a degree 2 extension over K, this means $\mathfrak{p} = \mathfrak{q}\overline{\mathfrak{q}}$ where $f(\mathfrak{q}|\mathfrak{p}) = f(\overline{\mathfrak{q}}|\mathfrak{p}) = 1$. By multiplicativity of inertial degrees in towers we have $f(\mathfrak{q}|p) = f(\mathfrak{p}|p)$ which is odd by hypothesis. On the other hand since $p \equiv 3 \pmod 4$, we know $p = \mathfrak{p}' \subset \mathbb{Q}(i)$ (i.e. p is inert), giving $f(\mathfrak{p}'|p) = 2$. But now, since $\mathfrak{q}$ must lie over $\mathfrak{p}'$, using multiplicativity of inertial degrees in towers again we have $f(\mathfrak{q}|p) = f(\mathfrak{q}|\mathfrak{p}')f(\mathfrak{p}'|p) = 2f(\mathfrak{q}|\mathfrak{p}')$ contradicting the fact that we already showed it was odd. □

An identical argument shows for any $\mathfrak{p} \subset K$ not lying over 3, $\mathfrak{p}$ splits in $K(\sqrt{-3})/K$ implies $N(\mathfrak{p}) \equiv 1 \pmod 3$.

Corollary 4.2. *Suppose $\mathfrak{p} \subset K$ splits in both $K(i)/K$ and $K(\sqrt{-3})/K$. Then one of the following holds*

- $\mathfrak{p}|2$ *and* $N(\mathfrak{p}) \equiv 1 \pmod 3$
- $\mathfrak{p}|3$ *and* $N(\mathfrak{p}) \equiv 1 \pmod 4$
- $N(\mathfrak{p}) \equiv 1 \pmod{12}$

For $K = \mathbb{Q}$, we have both 2 and 3 are ramified in $K(i)/K, K(\sqrt{-3})/K$ respectively, so eliminating 2, 3 torsion requires ramifying at primes $\mathfrak{p}, \mathfrak{q}$ such that $N(\mathfrak{p}) \equiv 1 \pmod 4, N(\mathfrak{q}) \equiv 1 \pmod 3$, necessarily eliminating the 12 in the denominator of Equation 2. We provide one last lemma relevant to eliminating 2/3-torsion in our algebras.

Lemma 4.3. *Suppose $\mathfrak{p}$ is a dyadic prime in a number field K, that splits in $K(i)/K$. Then $e(\mathfrak{p}|2) \equiv 0 \pmod 2$.*

Proof. As 2 ramifies in $\mathbb{Q}(i)$, we have that $e(\mathfrak{q}|2) \equiv 0 \pmod 2$ for any $\mathfrak{q} \subset K(i)/K$ lying over $\mathfrak{p}$. By assumption $\mathfrak{p} = \mathfrak{q}_1\mathfrak{q}_2$ in $K(i)/K$ giving $e(\mathfrak{q}_1|\mathfrak{p}) = e(\mathfrak{q}_2|\mathfrak{p}) = 1$ and by multiplicativity of e in towers it then must hold that $e(\mathfrak{p}|2) \equiv 0 \pmod 2$. □

Likewise, primes lying over 3 that split in $K(\sqrt{-3})/K$ must have even e, since 3 ramifies in $\mathbb{Q}(\sqrt{-3})$.

5. Dedekind Zeta Function Bounds and Computation

In order to understand the achievable quantities from 1, we must understand values of the Dedekind zeta function for arbitrary number fields. To do this, we first rewrite 1 in terms of $\zeta_K(-1)$ as the Siegel-Klingen Theorem [4, 10] tells us this value is always rational. Using the functional equation for the Dedekind zeta function we arrive at the following reformulation of Equation 1

$$\mathrm{Area}(\mathbb{H}^2/\mathcal{O}^1) = \frac{\pi\zeta_K(-1)}{2^{[K:\mathbb{Q}]-3}} \prod_{\mathfrak{p}\in \mathrm{Ram}_f(\mathscr{A})} (N(\mathfrak{p})-1)$$

giving

$$g-1 = \frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} \prod_{\mathfrak{p}\in \mathrm{Ram}_f(\mathscr{A})} (N(\mathfrak{p})-1) \tag{3}$$

We note that a prime $p \in \mathbb{Z}$ can only be achieved in 3 in two ways: Either $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}}$ is a rational number with p as its numerator and the term from the product is precisely equal to the denominator, or $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}}$ has numerator 1 and p is achieved by the terms of the product. Moreover, if $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{1}{n}$ for some $n \in \mathbb{N}$, and $p \neq 2^k - 1$, then since $N(\mathfrak{p})$ is a prime power and $p+1$ is even, we see that for any number field K, there is no prime $\mathfrak{p}$ such that $N(\mathfrak{p}) = p+1$. Hence for $p \neq 2^k - 1$, only strict multiplies of p, i.e. pk for $k > 1$, can arise from the product in 3 and in order to achieve $g - 1 = p$ we must have $k|n$.

We now present the bounds used in order to computationally search for Dedekind zeta function values of the above form.

Lemma 5.1. *For any number field K we have*

$$1 \le \zeta_K(2) \le \zeta(2)^{[K:\mathbb{Q}]}$$

where $\zeta(2)$ denotes the Riemann-zeta function evaluated at 2.

Proof. Using the product expression, $\zeta_K(2) \ge 1$ is trivial. Note

$$\begin{aligned}
\zeta_K(2) &= \prod_{\mathfrak{p}\subset\mathcal{O}_K} \frac{1}{1-\frac{1}{N(\mathfrak{p})^2}} \\
&= \prod_{p\in\mathbb{Z}}\prod_{\mathfrak{p}|p} \frac{1}{1-\frac{1}{N(\mathfrak{p})^2}} \\
&= \prod_{p\in\mathbb{Z}}\prod_{i=1}^{g} \frac{1}{1-\frac{1}{p^{2f_i}}}
\end{aligned}$$

where on the second line we index through all primes $p \in \mathbb{Z}$ and in the third line, g is the number of primes into which p splits and $f_i = f(\mathfrak{p}_i|p)$ for $1 \leq i \leq g$. We will show for each fixed prime $p \in \mathbb{Z}$

$$\left(\frac{1}{1-\frac{1}{p^2}}\right)^{[K:\mathbb{Q}]} \geq \prod_{\mathfrak{p}|p} \frac{1}{1-\frac{1}{N(\mathfrak{p})^2}}$$

By taking reciprocals and multiplying by $p^{2[K:\mathbb{Q}]}$, this is equivalent to showing

$$\left(p^2-1\right)^{[K:\mathbb{Q}]} \leq \prod_{i=1}^{g} \left(p^{2e_if_i} - p^{2e_if_i-2f_i}\right)$$

Claim: For any prime $p, (p^2-1)^k \leq p^{2k} - p^{2k-2}$, for all $k \in \mathbb{N}$.
Proof: We prove this by induction on k. The base case, $k = 1$ gives strict equality. Suppose the claim holds for k. Then

$$\begin{aligned}(p^2-1)^{k+1} &\leq (p^2-1)(p^{2k}-p^{2k-2}) \\ &= p^{2(k+1)} - p^{2k} - p^{2k} + p^{2k-2} \\ &\leq p^{2(k+1)} - p^{2k}\end{aligned}$$

as desired.
Now, we can rewrite

$$(p^2-1)^{[K:\mathbb{Q}]} = \prod_{i=1}^{g}(p^2-1)^{e_if_i}$$

Then for fixed i, setting $k = e_if_i$ we see from the claim

$$(p^2-1)^{e_if_i} \leq p^{2e_if_i} - p^{2e_if_i-2} \leq p^{2e_if_i} - p^{2e_if_i-2f_i}$$

which implies

$$(p^2-1)^{[K:\mathbb{Q}]} = \prod_{i=1}^{g}(p^2-1)^{e_if_i} \leq \prod_{i=1}^{g}\left(p^{2e_if_i} - p^{2e_if_i-2f_i}\right)$$

□

Using the functional equation to convert this inequality from an expression of $\zeta_K(2)$ to one of $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}}$, we obtain

$$\frac{2\Delta_K^{3/2}}{\pi^{2[K:\mathbb{Q}]}4^{[K:\mathbb{Q}]}} \leq \frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} \leq \frac{\Delta_K^{3/2}}{2^{[K:\mathbb{Q}]-1}12^{[K:\mathbb{Q}]}} \tag{4}$$

We also have the following bounds on the discriminant of a number field K from Takeuchi [11], originally due to Odlyzko [7].

Proposition 5.2. *Let K be a totally real number field of degree n. The following inequality holds*

$$\Delta_K > a^n e^{-b}$$

where a = 29.099, b=8.3185

Combining this bound on Δ_K based on the degree of K, with 4, we achieve that $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} \leq \frac{211}{2}$ only if $[K:\mathbb{Q}] \leq 11$ and the lower bound in 4 provides an upper bound on Δ_K in each degree which we provide in Table 2 in the appendix. Using these bounds we used Sage [12], specifically the work of Voight [13] and Jones–Roberts [2], to enumerate all number fields where $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} \leq \frac{211}{2}$, with $[K:\mathbb{Q}] \geq 3$, as we will address the quadratic case in a separate section. For those interested in the full table see this page. We simply provide a list of all number fields with $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{n}$ for some $n \in \mathbb{N}$, or $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{1}{n}$ where there exists some $k|n$ such that $211k+1$ is a prime power, in Table 1.

We now prove the impossibility of a genus 212 surface for a certain class of fields in Table 1.

Proposition 5.3. *There is no congruence surface of genus $g = 212$ constructed from a maximal order in a quaternion algebra over a totally real number field K where $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{1}{n}$ for some $n \in \mathbb{N}$.*

Proof. As discussed above, we must ramify at some prime $\mathfrak{p}$ such that $N(\mathfrak{p}) = 211k+1$ for some $k > 1, k|n$. The only $k|10, 30, 60, 84$ such that $211k+1$ is a prime power are $10, 42$. Since $4 \nmid 10, 42$, we have $211k+1 \equiv 3 \pmod 4$. Hence by Lemma 4.1 ramifying at such a prime does not eliminate 2 torsion. Moreover, in each of these fields $2 \nmid \Delta_K$ and hence by Lemmas 4.1 and 4.3 we must ramify at some $\mathfrak{q}$ such that $N(\mathfrak{q}) \equiv 1 \pmod 4$ in order to eliminate 2 torsion. This implies after ramifying at $\mathfrak{p}$ such that $N(\mathfrak{p}) = 211k+1$ and $\mathfrak{q}$ such that $N(\mathfrak{q}) = 4k'+1$ we have

$$g - 1 \geq \frac{(211k)4k'\zeta_K(-1)}{n} \prod_{\mathfrak{r} \in \mathrm{Ram}_f(\mathscr{A}) \setminus \{\mathfrak{p}, \mathfrak{q}\}} (N(\mathfrak{r}) - 1)$$

but since 4 is the highest power of 2 dividing n in each field, and $k = 10, 42$ is even, there is at least an extra factor of 2 left in the numerator, implying that after eliminating all torsion $g - 1 \geq 422$. □

6. The Quadratic Case

In this section we address separately the impossibility of a genus 212 congruence surface constructed from a maximal order in a quaternion algebra over a quadratic totally real number field K. In [14] Zagier showed for all quadratic fields $K \neq \mathbb{Q}(\sqrt{5})$, 12 clears the denominator of $\zeta_K(-1)$. Thus in our case, the denominator of $\frac{\zeta_K(-1)}{2}$ will always be a divisor of 24.

Proposition 6.1. *There is no congruence surface constructed from a maximal order in a quaternion algebra over a totally real quadratic number field K, where $\frac{\zeta_K(-1)}{2} = \frac{211}{n}$ for some $n|24$, $n \neq 12, 24$.*

Proof. Begin by noting, since $[K:\mathbb{Q}] = 2$, by construction, $\mathscr{A}$ over K must ramify at one of its infinite places and, by Theorem 2.8, this means that $|\mathrm{Ram}_f(\mathscr{A})|$ must be odd. If $\frac{\zeta_K(-1)}{2} = \frac{211}{n}$, then we must see that

$$\prod_{\mathfrak{p} \in \mathrm{Ram}_f(\mathscr{A})} (N(\mathfrak{p}) - 1) = n \tag{5}$$

in order to achieve a surface of genus 212. First suppose $\frac{\zeta_K(-1)}{2} = \frac{211}{6}$. Either

- $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ where $N(\mathfrak{p}) = 7$, $N(\mathfrak{q}_j) = 2$
- $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ where $N(\mathfrak{p}_1) = 4$, $N(\mathfrak{p}_2) = 3$, $N(\mathfrak{q}_j) = 2$

by Equation 5. In the former case, since $7 \equiv 3 \pmod 4$, by Lemma 4.1, $\mathfrak{p}$ does not split in $K(i)/K$. Thus we must have $\mathfrak{q}_j$ splits in $K(i)/K$ for some j, in order for our algebra to have no 2-torsion. But by Lemma 4.3 then $\mathfrak{q}_j$ must ramify in K, giving that $\mathfrak{q}_j$ is the only prime in K over 2. Hence $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}\}$ but this is not possible since $\mathrm{Ram}_f(\mathscr{A})$ must be of odd cardinality. If instead $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ where $N(\mathfrak{p}_1) = 4$, $N(\mathfrak{p}_2) = 3$, $N(\mathfrak{q}_j) = 2$, then $N(\mathfrak{p}_1) = 4$ implies 2 is inert in K. Thus $\mathfrak{p}_1$ is the only prime in K lying over 2 implying $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ but, again, as this set must have odd cardinality, this is not possible.

Next, suppose $\frac{\zeta_K(-1)}{2} = \frac{211}{4}$. Then by equation 5 we must see either

- $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ where $N(\mathfrak{p}) = 5$, $N(\mathfrak{q}_j) = 2$
- $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ where $N(\mathfrak{p}_1) = 3 = N(\mathfrak{p}_2) = 3$, $N(\mathfrak{q}_j) = 2$

Supposing the former case, simply note that by Lemma 4.1 no primes in $\mathrm{Ram}_f(\mathscr{A})$ split in $K(\sqrt{-3})/K$, and thus a ramification set of this form defines an algebra with 3-torsion. In the latter case, $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = 3$, implies there are two primes lying over 3 and by Lemma 4.3 these primes do not split in $K(\sqrt{-3})/K$, so again this ramification set defines an algebra with 3-torsion.

Now, suppose $\frac{\zeta_K(-1)}{2} = \frac{211}{3}$. Then we must see

$$\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\} \text{ where } N(\mathfrak{p}) = 4, N(\mathfrak{q}_j) = 2$$

In this case $N(\mathfrak{p}) = 4$ gives that 2 is inert with $\mathfrak{p}$ being the only prime lying over 2 giving $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}\}$. But by Lemma 4.3 $\mathfrak{p}$ does not split in $K(i)/K$ so this ramification defines an algebra with 2-torsion.

Finally suppose $\frac{\zeta_K(-1)}{2} = \frac{211}{2}$. Then we must see

$$\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\} \text{ where } N(\mathfrak{p}) = 3, N(\mathfrak{q}_j) = 2$$

By 4.1 $\mathfrak{p}$ does not split in $K(i)/K$. Hence there must be some $1 \leq j \leq n$ such that $\mathfrak{q}_j$ splits in $K(i)/K$. But by Lemma 4.3, such a $\mathfrak{q}_j$ must ramify implying there is only one prime lying over 2. This means $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}\}$ but this set must have odd cardinality, so no such $\mathscr{A}$ exists with this ramification set. □

Now using the bounds from Equation 4 we get that $\frac{\zeta_K(-1)}{2} > \frac{211}{12}$ whenever $\Delta_K > 574$. For quadratic fields, $\mathbb{Q}(\sqrt{d})$,

$$\Delta_K = \begin{cases} d, & d \equiv 1 \pmod 4 \\ 4d, & d \equiv 2, 3 \pmod 4 \end{cases}$$

Table 4 in the appendix lists all quadratic number fields up to $\mathbb{Q}(\sqrt{574})$ and the value of $\frac{\zeta_K(-1)}{2}$. Since none of these fields have numerator 211, this table along with Propositions 6.1 and 5.3 prove there is no congruence surface of genus 212 constructed from a maximal order in a quaternion algebra over a totally real quadratic number field.

7. The Different Ideal and the Discriminant

In the case where $[K : \mathbb{Q}] > 2$, an analysis like the one in the previous section is impossible as the splitting behavior of primes becomes much more complicated in arbitrary number fields. Instead we use results about the discriminant of a number field K to describe the splitting behavior of 2 and 3 in the fields in Table 1 in order to show the impossibility of a genus 212 surface in most of the remaining fields. We first define the different ideal. Given a number field K, we have a symmetric bilinear form on K given by $\langle x, y\rangle = tr(xy)$. Viewing K as a $\mathbb{Q}$-vector space, a lattice L in K is a $\mathbb{Z}$-module which is finitely generated by a $\mathbb{Q}$-span of K and the dual lattice $L^\vee$ is defined to be

$$L^\vee = \{x \in K | \langle x, y\rangle \in \mathbb{Z},\ \forall y \in L\}$$

Definition 7.1. *Let K be a number field. Its different ideal $\mathcal{D}_K$ is given by*

$$\mathcal{D}_K = {\mathcal{O}_K^\vee}^{-1} = \{x \in K | x\mathcal{O}_K^\vee \subset \mathcal{O}_K\}$$

We are particularly interested in the following properties of the different, one can find proofs of these properties in [6, Chapter 3, Section 2]

Theorem 7.2. *For any number field K*

$$N(\mathcal{D}_K) = \Delta_K$$

Theorem 7.3. *A prime $\mathfrak{p} \subset K$ is ramified if and only if $\mathfrak{p}|\mathcal{D}_K$. Moreover if a is the exact power which $\mathfrak{p}$ divides $\mathcal{D}_K$ and if $e = e(\mathfrak{p}|p)$ is the ramification index of $\mathfrak{p}$ over the unique prime $p \in \mathbb{Z}$ lies under $\mathfrak{p}$, then*

- $a = e - 1$ *if* $e \not\equiv 0 \pmod{p}$, *i.e.* $\mathfrak{p}$ *is tamely ramified*
- $a \geq e$ *if* $e \equiv 0 \pmod{p}$, *i.e.* $\mathfrak{p}$ *is wildly ramified*

Since the norm of an ideal is multiplicative this gives that if $\mathfrak{p}$ ramifies in K, then $N(\mathfrak{p})^a|\Delta_K$, where a is as above. From the analysis in the previous section we see the following.

Proposition 7.4. *In order to achieve a surface of genus 212 in the following fields we must see*

- *If* $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{6}$ *then either* $Ram_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ *where* $N(\mathfrak{p}) = 7$, $N(\mathfrak{q}_i) = 2$, *or* $Ram_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ *where* $N(\mathfrak{p}_1) = 4$, $N(\mathfrak{p}_2) = 3$, $N(\mathfrak{q}_i) = 2$
- *If* $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{4}$, *then* $Ram_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ *where* $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = 3$, $N(\mathfrak{q}_i) = 2$
- *If* $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{3}$ *then* $Ram_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ *where* $N(\mathfrak{p}) = 4, N(\mathfrak{q}_i) = 2$
- *If* $\frac{\zeta_K(-1)}{2^{K:\mathbb{Q}]-1}} = \frac{211}{2}$ *then* $Ram_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ *where* $N(\mathfrak{p}) = 3, N(\mathfrak{q}_i) = 2$

As some of these primes ramify in order to eliminate torsion in $\mathscr{A}$, we can use Theorem 7.3 to give conditions on Δ_K, in order for the ramification sets indicated above to be realized. First note in all of these cases, none of the primes $\mathfrak{p} \nmid 2$ split in $K(i)/K$ by Lemma 4.1, and thus Lemma 4.3 combined with 7.3 give $4|\Delta_K$ in order for any of the ramification sets above to define an algebra with no torsion. Moreover, in the case where $\frac{\zeta_K(-1)}{2} = \frac{211}{4}, \frac{211}{2}$, we must see that both 2 and 3 ramify in K, giving at least $36|\Delta_K$. Table 3 in the appendix provides a list of all fields from Table 2 where these further divisibility conditions on Δ_K hold.

In the case where $\frac{\zeta_K(-1)}{2} = \frac{211}{3}, \frac{211}{6}$, the ramification sets listed in Proposition 7.4 require some $\mathfrak{p} \in \mathrm{Ram}_f(\mathscr{A})$ lying over 2, which necessarily wildly ramifies by Lemma 4.3. Moreover, in each case it could hold that $N(\mathfrak{p}) = 2$, and since wild ramification gives only a lower bound on the power of $\mathfrak{p}$ dividing $\mathcal{D}$, no improvement can be made on the condition that $4|\Delta_K$. As a result, in the final section of this paper we analyze the fields in Table 3 computationally to show that the ramification sets in the Proposition 7.4 either define algebras with torsion, or simply do not define algebras, in each individual field.

8. Computations in Remaining Fields

To conclude, we present computations of the splitting behavior of 2 and 3 in the fields provided in Table 3. None of the methods from the preceding two sections can circumvent the need to check this computationally, as the possibility of wild ramification in these fields allows for no tighter restrictions on K and specifically Δ_K. We begin by analyzing the fields where $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{3}$

Proposition 8.1. *For the fields in Table 3, where $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{3}$, 2 factors as follows.*

Δ_K	$p(x)$	*Factorization*	*Norms*
13396	$x^3 - x^2 - 25x + 29$	$\mathfrak{p}^3$	$N(\mathfrak{p}) = 2$
1471216	$x^5 - 2x^4 - 7x^3 + 6x^2 + 8x - 4$	$\mathfrak{p}_1\mathfrak{p}_2^3$	$N(\mathfrak{p}_1) = 4, N(\mathfrak{p}_2) = 2$
1630076	$x^5 - 2x^4 - 9x^3 + 17x^2 + 4x - 12$	$\mathfrak{p}_1\mathfrak{p}_2^2$	$N(\mathfrak{p}_1) = 8, N(\mathfrak{p}_2) = 2$
1723364	$x^5 - 2x^4 - 7x^3 + 13x^2 + 8x - 11$	$\mathfrak{p}_1\mathfrak{p}_2^3$	$N(\mathfrak{p}_1) = 4, N(\mathfrak{p}_2) = 2$
17386832	$x^6 - 3x^5 - 4x^4 + 10x^3 + 6x^2 - 4x - 2$	$\mathfrak{p}_1\mathfrak{p}_2^5$	$N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = 2$
22340432	$x^6 - x^5 - 8x^4 + 5x^3 + 16x^2 - 5x - 1$	$\mathfrak{p}^3$	$N(\mathfrak{p}) = 4$
23556176	$x^6 - x^5 - 8x^4 + 5x^3 + 16x^2 - 7x - 7$	$\mathfrak{p}^3$	$N(\mathfrak{p}) = 4$

From Proposition 7.4 we see that only the four fields with a prime $\mathfrak{p}|2$ with $N(\mathfrak{p}) = 4$ could possibly give rise to a genus 212 surface. But, from Lemma 4.3, since none of these fields have a prime $\mathfrak{p}|2$ with even ramification index, none of the primes over 2 split in $K(i)/K$ giving that in these fields, the ramification sets defined in Proposition 7.4 define algebras with 2-torsion. Therefore, there is no construction of a genus 212 congruence surface constructed from a maximal order in a quaternion algebra over any of the fields in Proposition 8.1.

Proposition 8.2. *For the fields in Table 3, where $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{6}$, 2 factors as follows.*

Δ_K	$p(x)$	*Factorization*	*Norms*
1060708	$x^5 - 2x^4 - 7x^3 + 13x^2 + 10x - 17$	$\mathfrak{p}_1\mathfrak{p}_2^3$	$N(\mathfrak{p}_1) = 4, N(\mathfrak{p}_2) = 2$
12694016	$x^6 - 8x^4 - 2x^3 + 16x^2 + 8x - 1$	$\mathfrak{p}_1^2\mathfrak{p}_2^2$	$N(\mathfrak{p}_1) = 4, N(\mathfrak{p}_2) = 2$
15004240	$x^6 - 2x^5 - 11x^4 + 16x^3 + 35x^2 - 26x - 17$	$\mathfrak{p}^3$	$N(\mathfrak{p}) = 4$
15378496	$x^6 - 2x^5 - 5x^4 + 8x^3 + 6x^2 - 6x - 1$	$\mathfrak{p}^2$	$N(\mathfrak{p}) = 8$
17386832	$x^6 - 3x^5 - 4x^4 + 10x^3 + 6x^2 - 4x - 2$	$\mathfrak{p}_1\mathfrak{p}_2^5$	$N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = 2$
154050496	$x^7 - x^6 - 8x^5 + 6x^4 + 13x^3 - 9x^2 - x + 1$	$\mathfrak{p}_1\mathfrak{p}_2^2$	$N(\mathfrak{p}_1) = 2, N(\mathfrak{p}_2) = 8$

And 3 factors as follows.

Δ_K	$p(x)$	*Factorization*	*Norms*
1060708	$x^5 - 2x^4 - 7x^3 + 13x^2 + 10x - 17$	3	$N(3) = 729$
12694016	$x^6 - 8x^4 - 2x^3 + 16x^2 + 8x - 1$	3	$N(3) = 729$
15004240	$x^6 - 2x^5 - 11x^4 + 16x^3 + 35x^2 - 26x - 17$	$\mathfrak{p}_1\mathfrak{p}_2$	$N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = 27$
15378496	$x^6 - 2x^5 - 5x^4 + 8x^3 + 6x^2 - 6x - 1$	$\mathfrak{p}_1\mathfrak{p}_2$	$N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = 27$
17386832	$x^6 - 3x^5 - 4x^4 + 10x^3 + 6x^2 - 4x - 2$	3	$N(3) = 729$
154050496	$x^7 - x^6 - 8x^5 + 6x^4 + 13x^3 - 9x^2 - x + 1$	$\mathfrak{p}_1\mathfrak{p}_2$	$N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = 27$

As there are no norm 3 primes in these fields, we see by Proposition 7.4, in order to have a genus 212 congruence surface, $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ where $N(\mathfrak{p}) = 7$, $N(\mathfrak{q}_i) = 2$. In all but one field, every norm 2 prime has odd ramification index and thus by Lemma 4.3, these primes do not split in $K(i)/K$. Thus, in these fields, a ramification set of the aforementioned form defines an algebra with 2-torsion. The lone field where there exists a norm 2 prime with even ramification index is the field defined by $p(x) = x^6 - 8x^4 - 2x^3 + 16x^2 + 8x - 1$. Hence $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}\}$ where $N(\mathfrak{p}) = 7, N(\mathfrak{q}) = 2$ since there is only one norm 2 prime. But since this is a field of degree six,

$|\mathrm{Ram}_f(\mathscr{A})|$ must be odd and there is no such $\mathscr{A}$ with this ramification set. Therefore there is no construction of a genus $g = 212$ congruence surface constructed from a maximal order in quaternion algebra over any of the fields in Proposition 8.2. To conclude we consider the field defined by the polynomial $p(x) = x^4 - 20x^2 + 95$, which is such that $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{30}$.

Proposition 8.3. *In K defined by $p(x) = x^4 - 20x + 95$, in order for a surface of genus $g = 212$ to be achieved, we must see one of the following.*

(1) *$Ram_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$, where $N(\mathfrak{p}) = 31, N(\mathfrak{q}_i) = 2$*

(2) *$Ram_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$, where $N(\mathfrak{p}_1) = 16, N(\mathfrak{p}_2) = 3, N(\mathfrak{q}_i) = 2$*

(3) *$Ram_f(\mathscr{A}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$, where $N(\mathfrak{p}_1) = 11, N(\mathfrak{p}_2) = 4, N(\mathfrak{q}_i) = 2$*

In this field $2 = \mathfrak{p}_2$ where $N(\mathfrak{p}) = 4$. Thus in order for the first ramification set in Proposition 8.3 to be achieved, we must have $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}\}$ where $N(\mathfrak{p}) = 31$. Since $31 \equiv 3 \pmod 4$, by Lemma 4.3, this ramification set defines an algebra with 2-torsion. For the other two ramification sets we must have $\mathrm{Ram}_f(\mathscr{A}) = \{\mathfrak{p}, \mathfrak{q}\}$ since there are no norm 2 primes. But, as $[K : \mathbb{Q}] = 4$, we must have $|\mathrm{Ram}_f(\mathscr{A})|$ is odd and hence there is no such algebra with the second or third ramification set in Proposition 8.3. To conclude we recap our results from the previous sections to prove Theorem 1.3.

Proof of Theorem 1.3. To begin, Equation 2 provides the equation for $g - 1$ in the case where $K = \mathbb{Q}$. As $211k + 1$ for $k|12$ is not a prime power, it is not possible to achieve a genus 212 congruence surface when $K = \mathbb{Q}$.

Next, Proposition 5.3 and Section 6 prove there is no congruence surface constructed from a maximal order of genus 212 when $[K : \mathbb{Q}] = 2$.

For fields of higher degree, using Equation 4 and Proposition 5.2, we obtain an upper bound, U_n for Δ_K in each degree, such that when $\Delta_K > U_n$, $\frac{\zeta_K(-1)}{2^{n-1}} > \frac{211}{2}$. Then, using the enumerate all totally real number fields function from Voight, we compute all number fields where $\Delta_K < U_n$ in each degree, along with each field's value of $\frac{\zeta_K(-1)}{2^{n-1}}$, using Sage. Thus, Table 1 provides a list of all number fields where $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{211}{n}$ for some $n \in \mathbb{N}$, or $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{1}{n}$ where there exists some $k|n$ such that $211k + 1$ is a prime power, as these are the only number fields where the construction of a genus 212 surface is possible. However, Proposition 5.3 shows no such construction exists for K such that $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}} = \frac{1}{n}$ where there exists some $k|n$ such that $211k + 1$ is a prime power. Moreover, the results in Section 7 show no such construction exists for all fields in Table 2, except for those which we list in Table 4. To conclude, the results in Section 8 prove no such construction exists for all fields listed in Table 4. Thus, we have proved there is no congruence surface of genus 212, for any of the number fields listed in Table 2, which we have shown to be the only possible number fields where such a construction is possible. $\square$

9. Appendix

Δ_K	$p(x)$	$\frac{\zeta_K(-1)}{2}^{[K:\mathbb{Q}]-1}$
5	x^2-5	1/60
49	x^3-x^2-2x+1	1/84
7825	$x^3-25x-45$	211/6
9812	$x^3-x^2-19x+33$	211/4
13396	$x^3-x^2-25x+29$	211/3
725	$x^4-x^3-3x^2+x+1$	1/60
1125	$x^4-x^3-4x^2+4x+1$	1/30
2225	$x^4-x^3-5x^2+2x+4$	1/10
38000	x^4-20x^2+95	211/30
148889	$x^4-2x^3-15x^2+16x+51$	211/3
150057	$x^4-2x^3-7x^2+5x+7$	211/3
1060708	$x^5-2x4-7x^3+13x^2+10x-17$	211/6
1459417	$x^5-2x4-7x^3+11x^2+9x-2$	211/3
1471216	$x^5-2x^4-7x^3+6x^2+8x-4$	211/3
1630076	$x^5-2x^4-9x^3+17x^2+4x-12$	211/3
1723364	$x^5-2x^4-7x^3+13x^2+8x-11$	211/3
12694016	$x^6-8x^4-2x^3+16x^2+8x-1$	211/6
15004240	$x^6-2x^5-11x^4+16x^3+35x^2-26x-17$	211/6
15378496	$x^6-2x^5-5x^4+8x^3+6x^2-6x-1$	211/6
15700473	$x^6-12x^4-2x^3+39x^2+12x-19$	211/6
17386832	$x^6-3x^5-4x^4+10x^3+6x^2-4x-2$	211/3
17801408	$x^6-9x^4-4x^3+16x^2+14x+3$	211/4
18967381	$x^6-2x^5-8x^4+11x^3+20x^2-14x-17$	211/4
22340432	$x^6-x^5-8x^4+5x^3+16x^2-5x-1$	211/3
23556176	$x^6-x^5-8x^4+5x^3+16x^2-7x-7$	211/3
26768537	$x^6-x^5-11x^4+11x^2+x-2$	211/2
154050496	$x^7-x^6-8x^5+6x^4+13x^3-9x^2-x+1$	211/6
225111553	$x^7-2x^6-6x^5+9x^4+12x^3-8x^2-8x-1$	211/4
236583241	$x^7-x^6-9x^5+4x^4+16x^3-5x^2-5x+1$	211/3
343318749	$x^7-2x^6-6x^5+12x^4+8x^3-17x^2+3$	211/2

Table 1. Number Fields with Relevant $\frac{\zeta_K(-1)}{2}^{[K:\mathbb{Q}]-1}$ Values

$[K:\mathbb{Q}]$	Δ_K	$\Delta_K^{1/[K:\mathbb{Q}]}$
3	21922	27.99
4	254161	22.45
5	2946779	19.67
6	34165459	18.01
7	396120284	16.91
8	4592687600	16.13
9	53248420466	15.55
10	617371467322	15.10
11	7157912391165	14.74

Table 2. Upper Bounds for Discriminant in Fixed Degree

Δ_K	$p(x)$	$\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}}$
13396	$x^3 - x^2 - 25x + 29$	211/3
38000	$x^4 - 20x^2 + 95$	211/30
1060708	$x^5 - 2x4 - 7x^3 + 13x^2 + 10x - 17$	211/6
1471216	$x^5 - 2x^4 - 7x^3 + 6x^2 + 8x - 4$	211/3
1630076	$x^5 - 2x^4 - 9x^3 + 17x^2 + 4x - 12$	211/3
1723364	$x^5 - 2x^4 - 7x^3 + 13x^2 + 8x - 11$	211/3
12694016	$x^6 - 8x^4 - 2x^3 + 16x^2 + 8x - 1$	211/6
15004240	$x^6 - 2x^5 - 11x^4 + 16x^3 + 35x^2 - 26x - 17$	211/6
15378496	$x^6 - 2x^5 - 5x^4 + 8x^3 + 6x^2 - 6x - 1$	211/6
17386832	$x^6 - 3x^5 - 4x^4 + 10x^3 + 6x^2 - 4x - 2$	211/3
22340432	$x^6 - x^5 - 8x^4 + 5x^3 + 16x^2 - 5x - 1$	211/3
23556176	$x^6 - x^5 - 8x^4 + 5x^3 + 16x^2 - 7x - 7$	211/3
154050496	$x^7 - x^6 - 8x^5 + 6x^4 + 13x^3 - 9x^2 - x + 1$	211/6

Table 3. Remaining Number Fields where Splitting Behavior of 2,3 Must be Checked Computationally

d	$\frac{\zeta_K(-1)}{2}$	d	$\frac{\zeta_K(-1)}{2}$	d	$\frac{\zeta_K(-1)}{2}$	d	$\frac{\zeta_K(-1)}{2}$	d	$\frac{\zeta_K(-1)}{2}$
2	1/24	77	1	154	113/3	226	203/3	293	85/12
3	1/12	78	23/2	155	91/3	227	575/12	295	99
5	1/60	79	14	157	43/12	229	27/4	298	1165/12
6	1/4	82	27/2	158	89/3	230	311/6	299	238/3
7	1/3	83	43/4	159	35	231	62	301	31/3
10	7/12	85	3/2	161	16/3	233	53/6	302	479/6
11	7/12	86	155/12	163	467/12	235	141/2	303	87
13	1/12	87	13	165	11/3	237	35/6	305	41/3
14	5/6	89	13/6	166	503/12	238	67	307	1145/12
15	1	91	103/6	167	91/3	239	178/3	309	59/6
17	1/6	93	3/2	170	209/6	241	71/6	310	105
19	19/12	94	53/3	173	13/4	246	421/6	311	259/3
21	1/6	95	43/3	174	239/6	247	73	313	50/3
22	23/12	97	17/6	177	13/2	249	23/2	314	341/4
23	5/3	101	19/12	178	128/3	251	67	317	101/12
26	25/12	102	103/6	179	157/4	239	178/3	318	177/2
29	1/4	103	19	181	19/4	241	71/6	319	343/3
30	17/6	105	3	182	71/2	246	421/6	321	33/2
31	10/3	106	87/4	183	41	247	73	322	107
33	1/2	107	197/12	185	19/3	249	23/2	323	87
34	23/6	109	9/4	186	91/2	251	251/4	326	379/4
35	19/6	110	103/6	187	275/6	253	15/2	327	93
37	5/12	111	61/3	190	305/6	254	66	329	47/3
38	41/12	113	3	191	130/3	255	200/3	330	103
39	13/3	114	22	193	49/6	257	10	331	487/4
41	2/3	115	139/6	194	251/6	258	66	334	719/6
42	9/2	118	277/12	195	45	259	247/3	335	277/3
43	21/4	119	62/3	197	49/12	262	913/12	337	19
46	37/6	122	77/4	199	55	263	191/3	339	219/2
47	14/3	123	45/2	201	49/6	265	40/3	341	59/6
51	13/2	127	80/3	202	207/4	266	407/6	345	55/3
53	7/12	129	25/6	203	127/3	267	145/2	346	501/4
55	23/3	130	173/6	205	17/3	269	83/12	347	385/4
57	7/6	131	93/4	206	91/2	271	86	349	151/12
58	33/4	133	17/6	209	47/6	273	37/3	353	16
59	85/12	134	301/12	210	155/3	274	547/6	354	119
61	11/12	137	4	211	737/12	277	103/12	355	379/3
62	7	138	77/3	213	5	278	793/12	357	11
65	4/3	139	127/4	214	757/12	281	25/2	358	1505/12
66	28/3	141	3	215	139/3	282	235/3	359	109
67	41/4	142	63/2	217	29/3	283	1021/12	362	1159/12
69	1	143	73/3	218	583/12	285	8	365	65/6
70	67/6	145	16/3	219	173/3	286	563/6	366	253/2
71	29/3	146	83/3	221	16/3	287	72	367	382/3
73	11/6	149	35/12	222	105/2	290	449/6	370	791/6
74	41/4	151	37	223	60	291	269/3	371	334/3

d	$\frac{\zeta_K(-1)}{2}$	d	$\frac{\zeta_K(-1)}{2}$	d	$\frac{\zeta_K(-1)}{2}$
373	161/12	437	77/6	505	36
374	677/6	438	155	506	371/2
377	53/3	439	533/3	509	215/12
379	1781/12	442	517/3	510	194
381	77/6	443	569/4	511	232
382	413/3	445	55/3	514	674/3
383	322/3	446	440/3	515	363/2
385	71/3	447	147	517	127/6
386	249/2	449	51/2	518	174
389	151/12	451	1121/6	519	213
390	386/3	453	91/6	521	31
391	150	454	759/4	523	2563/12
393	43/2	455	448/3	526	1463/6
394	1907/12	457	30	527	532/3
395	346/3	458	1675/12	530	1087/6
397	57/4	461	61/4	533	37/2
398	665/6	462	481/3	534	1319/6
399	428/3	463	562/3	535	685/3
401	43/2	465	28	537	67/2
402	127	466	608/3	538	2647/12
403	899/6	467	1765/12	541	301/12
406	162	469	20	542	551/3
407	371/3	470	931/6	543	641/3
409	79/3	471	185	545	95/3
410	130	473	51/2	546	223
411	287/2	474	357/2	547	2849/12
413	73/6	478	593/3	551	601/3
415	491/3	479	160	553	119/3
417	47/2	481	101/3	554	2525/12
418	152	482	919/6	555	229
419	1619/12	483	169	557	233/12
421	209/12	485	101/6	559	775/3
422	1559/12	487	601/3	561	118/3
426	455/3	489	187/6	562	758/3
427	325/2	491	2119/12	563	761/4
429	16	493	119/6	565	151/6
430	1007/6	494	353/2	566	2497/12
431	142	497	27	569	109/3
433	163/6	498	560/3	570	680/3
434	433/3	501	20	571	1097/4
435	463/3	502	795/4	573	22
		503	481/3	574	270

TABLE 4. Quadratic Fields $\mathbb{Q}(\sqrt{d})$, and $\frac{\zeta_K(-1)}{2^{[K:\mathbb{Q}]-1}}$ for $d < 574$

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TRIVALENT PLANAR ALGEBRAS OVER FINITE FIELDS

EMI BRAWLEY

Abstract. In this paper, we study and classify trivalent planar algebras over finite fields, i.e., planar algebras generated by the Temperley-Lieb planar algebra and a trivalent vertex. Our results are that, where $\mathcal{C}_n$ is the space of diagrams with n boundary points, the trivalent planar algebras with $\dim \mathcal{C}_n$ bounded by 1, 0, 1, 1, 4, 10 for $0 \leq n \leq 5$ include the golden planar algebras, quantum $SO(3)$, and quantum G_2; this parallels the classification of trivalent planar algebras over $\mathbb{C}$, but the number of planar algebras in several of these families—in particular, the golden planar algebras–depends on the base field chosen. We also classify some trivalent planar algebras that arise when the nondegeneracy assumptions are weakened, and give directions for further investigation in this area.

Contents

Date: 27 July 2018.

1. Introduction

In this paper, we seek to generalize some past results regarding trivalent planar algebras, by weakening several of the assumptions made about the base field of these planar algebras and by weakening the nondegeneracy assumptions made in their definition.

The first goal of this paper is to extend some of the results of [MPS16] and [Kup94] to trivalent planar algebras over finite fields. We do this by following the calculations of the relations that define these planar algebras, performed over fields of nonzero characteristic rather than just over $\mathbb{C}$. Furthermore, where [MPS16] relies heavily on computer calculations and a strong non-degeneracy assumption regarding the inner product, our results depend on neither; all calculations were done by hand, using the methods described in [Kup94], and we assume very little about the inner product, relying only on the basic nondegeneracy of the 0-box space and the bigon relation.

We find that the dimensional assumptions that give rise to the trivalent planar algebras known as the golden planar algebras, the quantum $SO(3)$ family, and the quantum G_2 family result in trivalent planar planar algebras of very similar construction when taken over finite fields; we conjecture that the same holds true for the ABA class of planar algebras.

The second goal of this paper is to investigate the trivalent planar algebras that arise when some of the key nondegeneracy assumptions are weakened; in particular, we describe the results of weakening the nondegeneracy of the bigon relation. We find that the dimensional assumptions which give rise to the golden planar algebras result in a very different "faceless" trivalent planar algebra that is degenerate in this way, and we make some conjectures about the degenerate analogs of other nondegenerate trivalent planar algebras.

Finally, we propose some possible directions for future research to further the extension and generalization of past results regarding trivalent planar algebras.

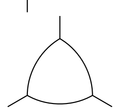
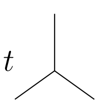
1.1. **Acknowledgments.** I would like to thank Professor Noah Snyder, who supervised this project, for his mentorship and support throughout the program, and Josh Edge and Patrick Chu for their mathematical advice and patience. I would also like to thank Professor Scott Morrison and Professor Emily Peters for for creating the figures used throughout this paper. Finally, I would like to thank Indiana University

Bloomington for hosting the REU program during which the work for this paper took place, as well as Professor Chris Connell, for organizing both the REU program and the 2018 Indiana Undergraduate Mathematics Research Conference, at which I presented the results of this project.

2. Trivalent planar algebras

In this section, we describe the notion of a trivalent planar algebra, as a special kind of planar algebra that is ”generated by the trivalent vertex,” along with an important theorem regarding the uniqueness of these planar algebras. We assume the reader has some familiarity with the notion of a planar algebra, and, in particular, is familiar with the various operations permitted by the action of the planar operad—rotation, capping, multiplication, and so forth. Throughout this paper, we will denote the n-box space of a planar algebra $\mathcal{V}$ as $\mathcal{V}_n$.

Definition 2.1. A trivalent planar algebra is a planar algebra $\mathcal{C}$ over some field $\mathbb{F}$; the elements of $\mathcal{C}_n$ are linear combinations of planar trivalent graphs with n univalent boundary points. A trivalent planar algebra has the following properties:

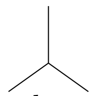
- $\dim \mathcal{C}_0 = 1$; $\bigcirc = d$
- $\dim \mathcal{C}_1 = 0$
- $\dim \mathcal{C}_2 = 1$;  $= b$ (1)
- $\dim \mathcal{C}_3 = 1$;  $= t$  (2)

A nondegenerate trivalent planar algebra is a trivalent planar algebra with $d \neq 0$ and $b \neq 0$; in a nondegenerate trivalent planar algebra, we normalize the trivalent vertex so that $b = 1$.

Where relevant, we refer to (1) as the “bigon relation” and (2) as the “triangle relation.”

We may think of a trivalent planar algebra as being generated by those diagrams which can be constructed from the trivalent vertex and the diagrams in the Temperley-Lieb planar algebra.

It is important to note that in [MPS16], trivalent planar algebras are defined over $\mathbb{C}$; our definition is slightly different, and has no restriction on the field chosen, as one of the primary goals of this paper is to extend the results of [MPS16] to trivalent planar algebras over finite fields.

Throughout this paper, we denote the set of trivalent planar graphs with n boundary points and at most k internal faces having four or more edges by $D(n, k)$. Furthermore, we refer to  as *a* or *the "trivalent vertex."*

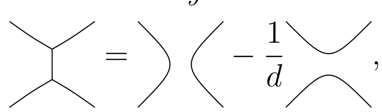
We now introduce a theorem concerning the uniqueness of nondegenerate trivalent planar algebras.

Theorem 2.2. *[MPS16, Corollary 2.4] Given a collection of linear relations amongst planar trivalent graphs, such that any closed diagram can be reduced to a multiple of the empty diagram by those relations, there there is a unique nondegenerate trivalent planar algebra satisfying those relations.*

This theorem allows us to define nondegenerate trivalent planar algebras solely in terms of the relations between their elements.

3. The golden planar algebras

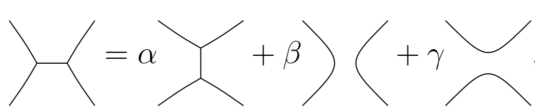
Theorem 3.1. *A trivalent planar algebra $\mathcal{C}$ over $\mathbb{F}$ with $\dim \mathcal{C}_4 = 2$ exists if and only if the quadratic $d^2 - d - 1 = 0$ has at least one solution in $\mathbb{F}$. If such a trivalent planar algebra exists, it is characterized by the relation*



with $d^2 - d - 1 = 0$; each solution to this quadratic corresponds to a unique trivalent planar algebra.

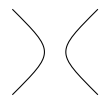
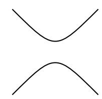
To prove this theorem, we require the following lemma, which describes an evaluation algorithm for closed diagrams in certain trivalent planar algebras.

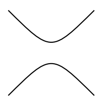
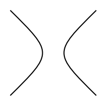
Lemma 3.2. *[cf. MPS16, Lemma 4.8] If there is a relation of the form*



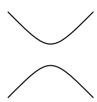
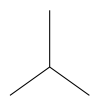
then any trivalent graph in $\mathcal{C}_n$ can be reduced to span $D(n, 0)$.

Proof. Suppose we have a diagram with some internal faces. Assuming there are no faces with three or fewer edges, and so we cannot immediately reduce the diagram to something with fewer faces, choose the smallest face. We can apply the relation above to the smallest face, yielding a sum of terms with either strictly fewer faces or a smaller smallest face. We may repeat this process until the diagram is reduced to a sum of terms with either strictly fewer faces or a smallest face with at most three edges; we may then, where appropriate, reduce that smallest face so that the diagram is reduced to a sum of terms which all have strictly fewer faces than the original diagram. By induction, therefore, we can write any diagram as a sum of terms with no internal faces. ☐

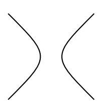
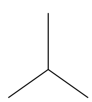
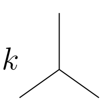
Proof of Theorem 3.1. Consider $\mathcal{C}_4$. Suppose that  and  are not linearly independent, i.e.,

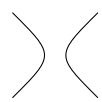
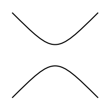
(3.1)  $= k \cdot$ 

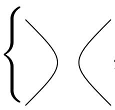
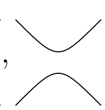
for some nonzero k. We have that

 $\cdot$  $= 0,$

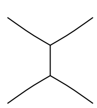
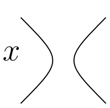
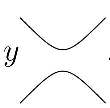
and

$k \cdot$  $\cdot$  $= k$  $\neq 0;$

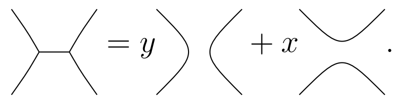
thus, in fact, we cannot have a relation like 3.1, and so  and  must be linearly independent. Since $\dim(\mathcal{C}_4) = 2$, the set

$\Bigg\{$  ,  $\Bigg\}$

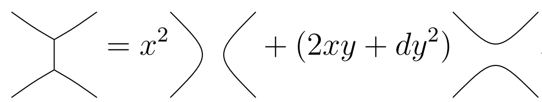
must form a basis for $\mathcal{C}_4$. Thus we must be able to express the other elements of $\mathcal{C}_4$ as linear combinations of these elements; in particular, we must have a relation of the form

 $= x$  $+ y$  .

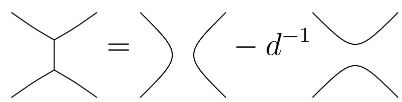
Rotating this relation gives



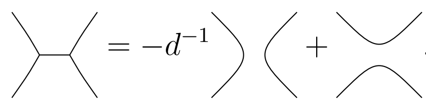
Capping the first relation gives that $x + dy = 0$, so $x = -dy$; this, together with our assumptions of nondegeneracy, implies that x and y must both be nonzero. Squaring the first relation and simplifying by the bigon relation gives that



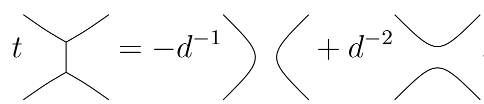
So we must have that $x = x^2$, and since $x \neq 0$ we can multiply by x^{-1} to give $x = 1$. Similarly, we have $y = 2xy + dy^2$, so $y = -d^{-1}$. Thus we have that



and

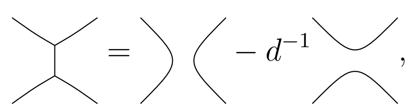


Multiplying these two relations together gives that



which in turn gives that $t = -d^{-1}$. Capping off the rotated relation gives that $d = d^2 - 1$.

By 3.2 we know that the resulting relation,

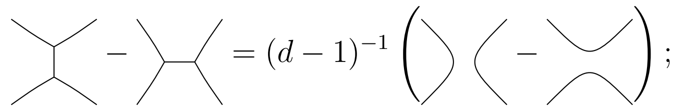


is sufficient to reduce any closed diagram to a multiple of the empty diagram, and so by 2.2 we have that, where such a relation is valid, this relation determines a unique nondegenerate trivalent planar algebra.

It may be true that if we are working over certain finite fields—for instance, the field with only two elements—there are no solutions to $d^2 - d - 1 = 0$, in which case it is necessary to work instead over some extension of our original field. In fields of characteristic other than 2, we have that $d = 2^{-1}(1 \pm \sqrt{5})$, yielding that $t = 2^{-1}(1 \mp \sqrt{5})$. Thus we have two possible values of d and t; the only exception is in fields of characteristic 5, where we must have that $d = t = 3$.

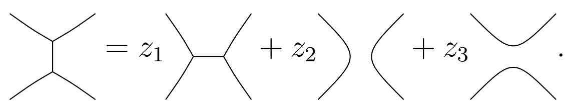
4. Quantum SO(3)

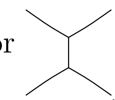
Theorem 4.1. *If $\mathcal{C}$ is a trivalent planar algebra, and $\dim \mathcal{C}_4 = 3$, then $\mathcal{C}$ is a member of the one-parameter family of quantum $SO(3)$ planar algebras, characterized by relations of the form*

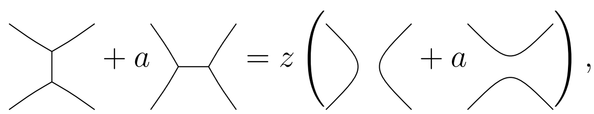


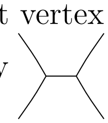
each value of d corresponds to a unique nondegenerate trivalent planar algebra.

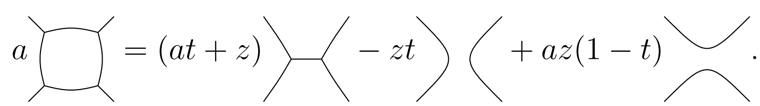
Proof. Since $\dim \mathcal{C}_4 = 3$, we must have some relation among the elements of $D(4,0)$, and in fact this relation can be written in the form



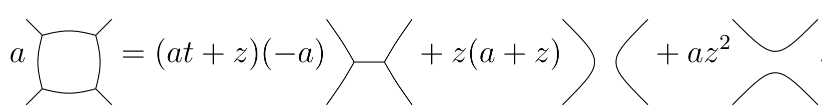
Rotating and substituting this relation in for  gives that $1 = z_1^2$, $z_1 z_2 + z_3 = 0$, $z_1 z_3 + z_2 = 0$. Thus we see that in fact we can rewrite this relation as



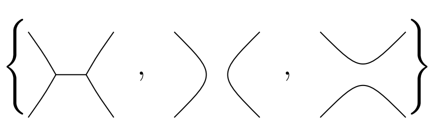
with $a^2 = 1$. Capping the relation gives that $a = z(1 + da)$, i.e., $z = (a+d)^{-1}$; we also see from this relation that $z \neq 0$. Multiplying the relation by a trivalent vertex gives that $z = at + 1$, i.e., $t = a(a+d)^{-1} - a$. Now, multiplying the relation by  gives a relation for the square in terms of the basis elements of $\mathcal{C}_4$:



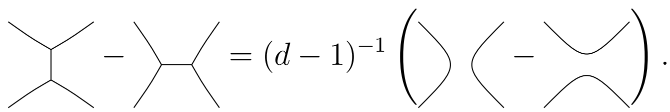
Rotating this relation gives yet another relation for the square:



Since



is a basis for $\mathcal{C}_4$, we then have that $(at + z)(-a) = at + z$, $z(a + z) = -zt$, and $az^2 = az(1 - t)$; solving this system of equations gives that $a = -1$. Thus we have that $t = 1 - (d-1)^{-1}$, and we have the relation



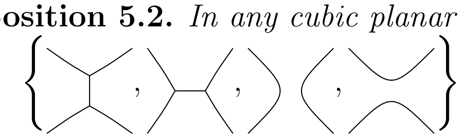
From 3.2 and 2.2, we have that each relation of this form determines a unique nondegenerate trivalent planar algebra.

∎

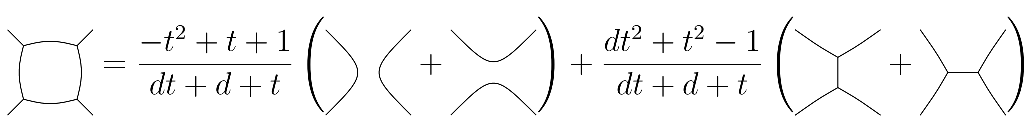
5. Cubic planar algebras

Definition 5.1. A *cubic planar algebra* is a trivalent planar algebra where $\dim \mathcal{C}_4 = 4$.

Proposition 5.2. *In any cubic planar algebra,*

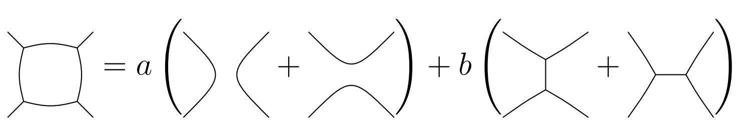
(a)  *form a basis of $\mathcal{C}_4$. [MPS16, Proposition 4.16]*

(b) we have the square relation



.

Proof. We begin by writing the square as a linear combination of the basis elements of $\mathcal{C}_4$; since the square is rotationally invariant, we can write



Multiplying by a trivalent vertex gives the equation $t^2 = a + b + bt$; capping the relation gives that $a + da + b = 1$. Assuming that $dt + d + t \neq 0$, this system of equations gives exactly

$$a = \frac{-t^2+t+1}{dt+d+t}, b = \frac{dt^2+t^2-1}{dt+d+t}.$$

∎

6. *ABA*

Over $\mathbb{C}$, the ABA family of trivalent planar algebras is a family of cubic planar algebras with $\dim \mathcal{C}_5 = 8$ [MPS16, cf. Section 5]. The proof of the existence of the relations among the elements of $\mathcal{C}_5$ which define these planar algebras relies heavily on the assumption of nondegeneracy of the inner product; we have so far been unable to reproduce these relations by any other method, but we conjecture that it is possible.

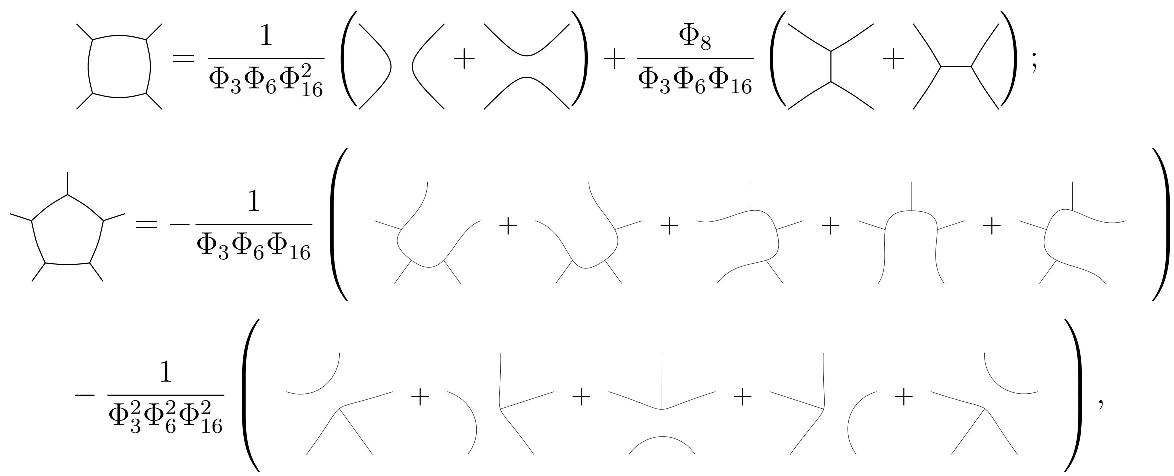
Conjecture 1. *A nondegenerate cubic trivalent planar algebra $\mathcal{C}$ over $\mathbb{F}$ with $\dim \mathcal{C}_5 = 8$ exists if and only if the equation $\omega^5 - 1 = 0$ has multiple solutions; in such a planar algebra, we have $t^2 - t - 1 = 0$.*

Furthermore, we conjecture that the description above minimizes $\dim \mathcal{C}_5$ for a cubic planar algebra.

Conjecture 2. *No nondegenerate cubic trivalent planar algebra $\mathcal{C}$ exists such that $\mathcal{C}_5 < 8$.*

7. Quantum G_2

Theorem 7.1. *If $\mathcal{C}$ is a cubic planar algebra, and $\dim \mathcal{C}_5 = 10$, then $\mathcal{C}$ is a quantum G_2 planar algebra, defined by the relations $d = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10}$; $t = -\Phi_12/\Phi_{16}$;*

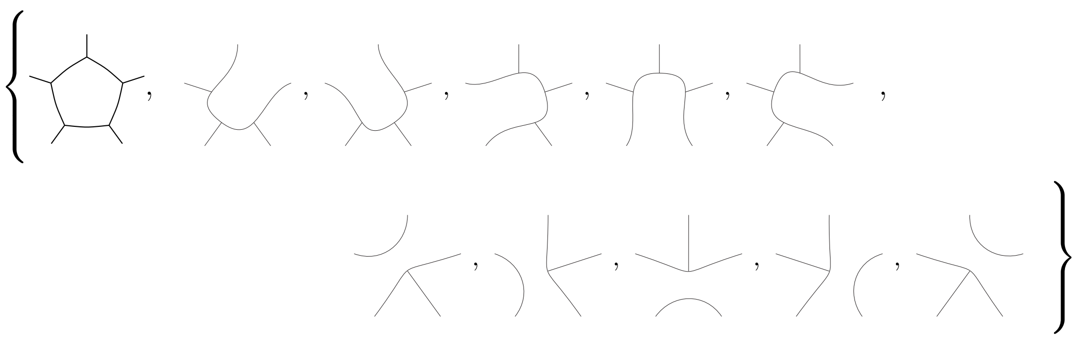


where Φ_k is the kth cyclotomic polynomial, i.e, $\Phi_k = \prod_\zeta (q^{1/2} - \zeta q^{-1/2})$ where the product is taken over all primitive kth roots of unity; q must satisfy

$$\Phi_3\Phi_6\Phi_7\Phi_{14}\Phi_{16}\Phi_{24} \neq 0.$$

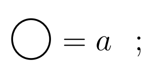
In order to prove this theorem, we require the following lemma.

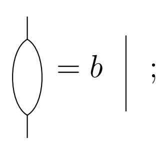
Lemma 7.2. *[MPS16, Lemma 5.13, Lemma 5.15] Suppose $\mathcal{C}$ is a planar algebra generated by a trivalent vertex, with relations reducing n-gons for each $n \leq 4$. Suppose further there is some relation between the diagrams*

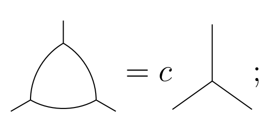


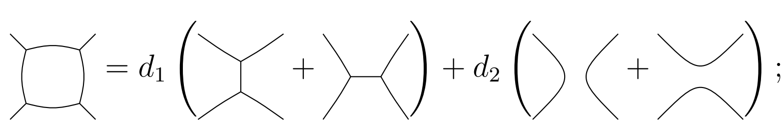
Then there is a relation reducing the pentagon (as a linear combination of the acyclic diagrams in $\mathcal{C}_5$), and $\mathcal{C}_5$ is spanned by the acyclic diagrams in $\mathcal{C}_5$.

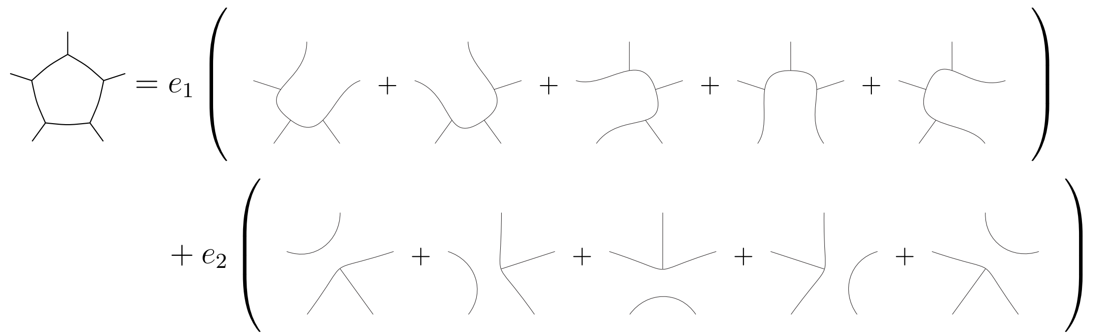
Proof of Theorem 7.1. There are exactly 10 acyclic diagrams in $\mathcal{C}_5$, so the assumption that $\dim \mathcal{C}_5 = 10$ together with the above lemma gives that these acyclic diagrams form a basis for $\mathcal{C}_5$. We can now express our assumptions as the following set of equations:











Note that we have not yet, as in other calculations, normalized the trivalent vertex so that $b = 1$.

We now proceed as in [Kup94], constructing diagrams with exactly two faces, each of which has five or fewer edges, and simplifying these diagrams in two ways (varying which face is simplified first). This yields the following system of equations.

$$\begin{aligned}
b^2 &= bd_1 + d_2 + ad_2 \\
c^2 &= bd_1 + cd_1 + d_2 \\
bc &= 2be_1 + 2e_2 + ae_2 + ce_1 \\
cd_1 &= d_1e_1 + ce_1 + be_1 + e_2 \\
cd_2 &= d_2e_1 + be_2 \\
d_1e_1 + d_2 &= e_1^2 \\
d_1e_1 + d_1^2 &= e_1^2 + ce_1 + d_1e_1 \\
d_1e_1 &= e_1^2 + d_1e_1 + e_2 \\
d_1e_2 + cd_2 &= e_1e_2 + be_2 + 2d_2e_1 \\
d_1e_2 &= e_1e_2 + d_2e_1 \\
d_1e_2 + d_1d_2 &= e_1e_2 + ce_2
\end{aligned}$$

Setting $e_1 = 1$ and $d_1 = -q^2 - q^{-2}$ gives a unique set of solutions $a = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10}$, $b = -q^6 - q^4 - q^2 - q^{-2} - q^{-4} - q^{-6}$, $c = q^4 + 1 + q^{-4}$, $d_2 = q^2 + 1 + q^{-2}$, $e_2 = -1$. This is identical to the solution obtained in [Kup94], except for the choice of q^2 instead of q; the calculations involved are the same in a field of nonzero characteristic as they are over the complex numbers. As Kuperberg notes, we can multiply b, c, d_1, and e_1 by a constant, and d_2 and e_2 by the square of that constant to obtain another solution; we let this constant be $-\Phi_3\Phi_6\Phi_{16}$, so that $b = 1$, and thus obtain the coefficients given in the statement of Theorem 7.1. The restrictions on q come from the nondegeneracy assumptions that $a \neq 0$ and $b \neq 0$.

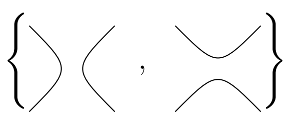
8. Abandoning nondegeneracy

Recall that in the definition of a trivalent planar algebra, we assumed that b, the coefficient in the bigon relation, was nonzero, and we normalized this to give $b = 1$. It is interesting to investigate what happens if we weaken this nondegeneracy assumption, allowing $b = 0$.

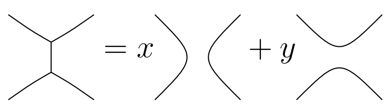
8.1. **Degenerate "golden" planar algebras.**

Theorem 8.1. *Suppose $\mathcal{C}$ is a trivalent planar algebra with $\dim \mathcal{C}_0 = 1$, $\dim \mathcal{C}_1 = 1$, $\dim \mathcal{C}_2 = 1$ with $b = 0$, $\dim \mathcal{C}_3 = 1$, and $\dim \mathcal{C}_4 = 2$. Then the elements of $\mathcal{C}_n$ are linear combinations of non-crossing pair and triple partitions of a circle with n boundary points; in particular, this planar algebra is "faceless", i.e., any diagram with an interior face is equal to zero.*

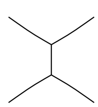
Proof. As in Section 3, we begin from the fact that



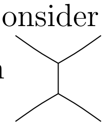
forms a basis for $\mathcal{C}_2$, and thus

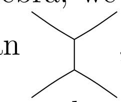


for some x and y. Capping this relation gives that $x = -dy$; squaring the relation gives that $x^2 = 0$ and $2xy + dy^2 = 0$. Solving this system of equations gives that $x = y = 0$, and thus we have that

 $= 0.$

Unlike in the case of the golden planar algebras, this relation gives no restrictions on d; thus we have a one-parameter family of these degenerate planar algebras.

Now suppose we have a diagram with an interior face. Consider this interior face. If the face has at least three edges, the diagram contains an  and is thus equal to zero; if the face has one or two edges, we know already that we may reduce the diagram to zero. Thus any diagram with faces must be equal to zero.

Suppose we have a diagram in $\mathcal{C}_n$ with a connected component that contains more than one trivalent vertex. Since this is a trivalent planar algebra, we must have at least two trivalent vertices connected to each other, forming an , and thus the diagram must be equal to zero. Thus we see that each connected component of a nonzero element of $\mathcal{C}_n$ can contain at most one trivalent vertex. We can therefore characterize the nonzero elements of $\mathcal{C}_n$ as linear combinations of non-crossing pair and triple partitions of a circle with n boundary points. This also immediately gives some bounds on the dimensions of the $\mathcal{C}_n$; for instance, $\dim \mathcal{C}_5 \leq 5$, and $\dim \mathcal{C}_6 \leq 8$. □

Directions for further investigation. It would be interesting to investigate the degenerate trivalent planar algebras that arise from the dimension assumptions that produced the other nondegenerate planar algebras described in this paper. To that end, we make the following conjecture.

Conjecture 3. *Suppose $\mathcal{C}$ is a trivalent planar algebra with $b = 0$, and $\dim \mathcal{C}_4 = 3$. Then $\mathcal{C}$ is, as described above, a "faceless" planar algebra—that is, its only nonzero elements are linear combinations of diagrams with no internal faces.*

It would also be interesting to investigate the degenerate trivalent planar algebras that arise when $\dim \mathcal{C}_4 = 4$, $\dim \mathcal{C}_4 = 5$, and so on. In fact, this kind of degeneracy would seem to make describing these planar algebras very easily due to the ease of computing relations, although these planar algebras might perhaps be more difficult to classify. One could also investigate degenerate and nondegenerate braided trivalent planar algebras and their skein theoretic invariants over fields of nonzero characteristic.

References

[Kup94] Greg Kuperberg. "The quantum G2 link invariant". In: *International Journal of Math* (1994), pp. 61–85.

[MPS16] S. Morrison, E. Peters, and N. Snyder. "Categories generated by a trivalent vertex". In: *ArXiv e-prints* (July 2016). arXiv: `1501.06869 [math.QA]`.

VOLUME ESTIMATES OF IDEAL HYPERBOLIC SIMPLICES

CORNELL HOLMES

Abstract. In the paper "Simplices of maximal volume in hyperbolic n-space," U. Haagerup and H. Munkholm estimate the volume of the ideal regular hyperbolic n-simplex by relating it to the volume of the regular euclidean n-simplex inscribed in the unit sphere. In this paper, we use analogous techniques and the Lasserre-Avrachenkov theorem to estimate the volumes of ideal hyperbolic n-simplices by comparing the volumes of ideal hyperbolic n-simplices to the volumes of the euclidean n-simplices with the same vertices as the hyperbolic simplices in the projective model.

1. Introduction

1.1. **Preliminaries.** An n-simplex is ideal if all of its vertices are on the boundary of $\mathbf{H}^n$. Throughout this paper, let $\tau[n]$ denote any ideal hyperbolic n-simplex with vertices $v_0, \ldots, v_n$ in the projective model of $\mathbf{H}^n$, let $\sigma[n]$ be the euclidean n-simplex with the same vertices on S^{n-1}, and let φ_n be the function

$$\varphi_n(\alpha) = \int_{\sigma[n]} (1-r^2)^{-\alpha} d\mathbf{r}, \qquad \alpha < n.$$

The approach to the conjecture is motivated by the following observations from Haagerup and Munkholm in [1].

Remark 1. *The volume $Vol(\tau[n])$ of $\tau[n]$ can be expressed as an integral over $\sigma[n]$ as*

$$Vol(\tau[n]) = \int_{\sigma[n]} (1-r^2)^{-(n+1)/2} d\mathbf{r}. \tag{1}$$

Remark 2. *φ_n is logarithmically convex as it is the integral of a logarithmically convex function. Thus*

$$\left(\frac{\varphi_n(0)}{\varphi_n(-1)}\right)^{\frac{n-1}{2}} \frac{\varphi_n(\frac{n+1}{2})}{\varphi_n(\frac{n-1}{2})} \leq \left(\frac{\varphi_n(\frac{n+1}{2})}{\varphi_n(0)}\right) \leq \left(\frac{\varphi_n(\frac{n+1}{2})}{\varphi_n(\frac{n-1}{2})}\right)^{\frac{n+1}{2}} \tag{2}$$

holds for $n \geq 2$.

1.2. **Conjecture.** We are interested in studying the quantity $Vol(\tau[n]) = \varphi_n(\frac{n+1}{2})$ by obtaining precise bounds for $\varphi_n(\frac{n+1}{2})/\varphi_n(0)$. $\varphi_n(0) = Vol(\sigma[n])$ is easily computable by a determinant formula given the vertices, so we need only study the ratios $\varphi_n(\frac{n+1}{2})/\varphi(\frac{n-1}{2})$ and $\varphi_n(0)/\varphi_n(-1)$ to use (2) to estimate $Vol(\tau[n])$. Let $\{\tau[n]\}_{n=1}^{\infty}$ be any sequence of ideal hyperbolic n-simplices in n with corresponding euclidean n-simplices $\{\sigma[n]\}_{i=1}^{\infty}$. We may assume that the corresponding euclidean simplex of $\tau[n]$ has a center of mass at zero by isometry of $\mathbf{H}^n$ for each n. That is, $\sum_{i=0}^{n} v_i = 0$. For any such sequence we conjecture that

$$\lim_{n\to\infty} \frac{Vol(\tau[n])}{Vol(\sigma[n])} = \sqrt{e}. \tag{3}$$

2. Proof

2.1. **Lower Bound.** First, we determine $\varphi_n(0)/\varphi_n(-1)$ given any euclidean simplex $\sigma[n]$ corresponding to $\tau[n]$.

Theorem 1. *Let $\sigma[n]$ be any euclidean n-simplex with vertices $v_0, \ldots, v_n$ on S^{n-1} for $n \geq 1$. If $c = ||\frac{1}{n+1}\sum_{i=0}^{n} v_i||$, then*

$$Vol(\sigma[n])^{-1} \int_{\sigma[n[} (1 - r^2) d\mathbf{r} = (1 - c^2)\frac{n+1}{n+2}. \tag{4}$$

Proof. Let $\Delta[n] = \{(t_0, \ldots, t_n) | t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$ be the standard n-simplex in $\mathbb{R}^{n+1}$ and let S be the set of permutations of $0, \ldots, n$. Define E to be the formation of mean values over all permutations $\pi \in S$ such that $E(f) = 1/(n+1)! \sum_{\pi \in S} f(\pi)$ for any function f of permutations. The inclusion function from (0,1] to $\mathbb{R}^n$ is continuous and concave, so our integral becomes

$$\begin{aligned} Vol(\sigma[n])^{-1} \int_{\sigma[n]} (1 - r^2) d\mathbf{r} &= \int_{\Delta[n]} (1 - ||\sum_{i=0}^{n} t_i v_i||^2) d\mu \\ &= E(\int_{\Delta[n]} (1 - ||\sum_{i=0}^{n} t_{\pi(i)} v_i||^2) d\mu) \end{aligned}$$

for μ the measure on $\Delta[n]$ such that $\mu(\Delta[n])$=1 by the construction in Lemma 3 of [1]. Note that

$$E(1 - ||\sum_{i=0}^{n} t_{\pi(i)} v_i||^2) = \frac{n+1}{n}(1 - c^2)(1 - \sum_{i=0}^{n} t_i^2).$$

It follows by linearity of the integral and the integrand that

$$E(\int_{\Delta[n]} (1 - ||\sum_{i=0}^{n} t_{\pi(i)} v_i||^2) d\mu) = \int_{\Delta[n]} E(1 - ||\sum_{i=0}^{n} t_{\pi(i)} v_i||^2) d\mu$$
$$= (1 - c^2) \left[\frac{n+1}{n} \int_{\Delta[n]} (1 - \sum_{i=0}^{n} t_i^2) d\mu \right].$$

Lemma 5 in [1] shows

$$\frac{n+1}{n} \int_{\Delta[n]} (1 - \sum_{i=0}^{n} t_i^2) d\mu = \frac{n+1}{n+2},$$

therefore we have

$$Vol(\sigma[n])^{-1} \int_{\sigma[n]} (1 - r^2) d\mathbf{r} = (1 - c^2) \frac{n+1}{n+2}.$$

□

2.2. **Upper Bound.** In our initial investigation of the ratio $\varphi_n(\frac{n+1}{2})/\varphi_n(\frac{n-1}{2})$ we generalized a geometric approach explored in [1].

Theorem 2. *For each $i = 0, \ldots, n$ let c_i be the signed euclidean distance between the affine $(n-1)$-plane A_i that contains the i^{th} boundary face $\partial_i \sigma[n]$ of $\sigma[n]$ and the origin, positive if the outward normal points away from the origin and negative if the outward normal points towards the origin. Also for each $i = 0, \ldots, n$ let $\tau_i[n]$ be ideal hyperbolic n-simplex viewed in the upper half space model with n vertices on the $(n-2)$-sphere $S^{n-2} = \{x \in \mathbb{R}^n | x_n = 0, ||x|| = 1\}$ corresponding to the vertices of any euclidean $(n-1)$-simplex similar to $\partial_i \sigma[n]$ on S^{n-2} and the last vertex at the point at ∞. Then for $n \geq 2$*

$$\frac{\varphi(\frac{n-1}{2})}{\varphi(\frac{n+1}{2})} = (n-1) \left[\sum_{i=0}^{n} c_i \frac{Vol(\tau_i[n])}{Vol(\tau[n])} - 1 \right].$$

Proof. Define the vector field

$$\mathbf{V}(\mathbf{r}) = (1 - r^2)^{-(n-1)/2} \mathbf{r}$$

on $\sigma[n]$ and let $\mathbf{n}$ be the outward normal to the boundary $\partial\sigma[n]$. Along each boundary face $\partial_i \sigma[n]$, the dot product $\mathbf{r} \cdot \mathbf{n}$ is equal to the length of the orthogonal vector between A_i and the origin up to a sign that depends on the orientation of $\mathbf{n}$ and is therefore constant. That is, $\mathbf{r} \cdot \mathbf{n} = c_i$.

For each $\partial_i\sigma[n]$, let ρ_i be the radius of the $(n-2)$-sphere circumscribed around $\partial_i\sigma[n]$ that lays in A_i, and let $\mu_\mathbf{i}$ be the vector from the origin to the circumcenter. Define the function

$$\beta_i : \partial_i\sigma[n] \to \mathbb{R}, \qquad \beta_i(\mathbf{r}) = ||\mathbf{r} - \mu_\mathbf{i}||.$$

Note that $\mu_\mathbf{i}$ is orthogonal to the linear plane corresponding to A_i, so for all $\mathbf{r} \in \partial_i\sigma[n]$

$$\begin{aligned}\mathbf{r}\cdot\mathbf{r} &= ((\mathbf{r}-\mu_\mathbf{i}) + \mu_\mathbf{i})\cdot((\mathbf{r}-\mu_{\mathbf{i}_i}) + \mu_\mathbf{i})\\ &= \beta_i(\mathbf{r})^2 + \mu_\mathbf{i}\cdot\mu_\mathbf{i}\\ &= \beta_i(\mathbf{r})^2 + (1-\rho_i^2)\end{aligned}$$

The last equality comes from expanding the inner product of a vertex v_j of $\partial_i\sigma[n]$ with itself. Thus

$$\begin{aligned}\int_{\partial\sigma[n]} \mathbf{V}\cdot\mathbf{n}dS &= \sum_{i=0}^{n}\int_{\partial_i\sigma[n]}\mathbf{V}\cdot\mathbf{n}dS\\ &= \sum_{i=0}^{n} c_i\int_{\partial_i\sigma[n]}(1-r^2)^{-(n-1)/2}dS\\ &= \sum_{i=0}^{n} c_i\int_{\partial_i\sigma[n]}(\rho_i^2-\beta_i^2)^{-(n-1)/2}dS.\end{aligned}$$

Fix any $i = 0, \ldots, n$. By an isometry in the projective model, we can assume that $A_i = \{x \in \mathbb{R}^n | x_n = b_i\}$ for some $|b_i| < 1$. Now define the function

$$f_i : A_i \to \mathbb{R}^{n-1} \qquad f_i(x) = \frac{1}{\rho_i}(x_1, \ldots, x_{n-1}, 0)$$

and let $f_i(\partial_i\sigma[n]) = \sigma_i[n-1]$. This transforms our integral over $\partial_i\sigma[n]$ into the integral

$$\int_{\sigma_i[n-1]}(\rho_i^2 - \beta_i^2 r^2)^{-(n-1)/2}\rho_i^{n-1}dr = \int_{\sigma_i[n-1]}(1-r^2)^{-(n-1)/2}dr.$$

$\sigma_i[n-1]$ is a euclidean $(n-1)$-simplex with vertices on S^{n-2} similar to $\partial_i\sigma[n]$ so we can take $\tau_i[n]$ as the hyperbolic n-simplex that projects onto the ideal hyperbolic $(n-1)$-simplex with vertices corresponding to those of $\sigma_i[n-1]$ on S^{n-2} from the point ∞ in the upper half plane model. It follows from (2.2) in [1] that

$$\int_{\partial\sigma[n]}\mathbf{V}\cdot\mathbf{n}dS = (n-1)\sum_{i=0}^{n} c_i Vol(\tau_i[n]).$$

The result follows from equation (1) and the divergence theorem.

$$\begin{aligned}
\int_{\sigma[n]} div\mathbf{V}(r)dr &= \int_{\sigma[n]} (1-r^2)^{-(n-1)/2}dr + (n-1)\int_{\sigma[n]} (1-r^2)^{-(n+1)/2}dr \\
&= \int_{\sigma[n]} (1-r^2)^{-(n-1)/2}dr + (n-1)Vol(\tau[n]) \\
&= (n-1)\sum_{i=0}^{n} c_i Vol(\tau_i[n]).
\end{aligned}$$

□

2.3. **Lassere-Avrechenkov Method.** Our second approach to studying the upper bound we attempted to turn our integral over a euclidean simplex into a power series. For any real-valued symmetric multilinear q-form $H : (\mathbb{R}^n)^q \to \mathbb{R}$ the Lasserre-Avrachenkov theorem shown in [2] gives

$$\int_{\sigma[n]} H(\mathbf{x},\ldots,\mathbf{x})dx = \frac{Vol(\sigma[n])}{\binom{n+q}{q}} \sum_{0\le i_1\le i_2,\ldots,\le i_q\le n} H(v_{i_1},\ldots,v_{i_q}).$$

To compare $\varphi_n(\frac{n+1}{2})$ to $\varphi_n(\frac{n-1}{2})$, we consider the integrand $(1-r^2)^{-\alpha}$. Expanding this as a power series in terms of r we have

$$(1-r^2)^{-\alpha} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} r^{2k}, \qquad |r|<1.$$

Each term r^{2k} is a homogenous polynomial of degree $2k$, so there exists a unique multilinear symmetric $2k$-form H_{2k} for

$$H_{2k}(\mathbf{x_1},\ldots,\mathbf{x_{2k}}) = \frac{1}{(2k)!}\sum_{\pi\in S_{2k}}\prod_{l=1}^{k}\langle \mathbf{x}_{\pi(2l-1)},\mathbf{x}_{\pi(2l)}\rangle$$

such that for all $\mathbf{x}\in\sigma[n]$

$$||\mathbf{x}||^{2k} = H_{2k}(\mathbf{x},\ldots,\mathbf{x}).$$

Therefore we can express $\varphi_n(\alpha)$ as

$$\begin{aligned}\int_{\sigma[n]}(1-r^2)^{-\alpha}d\mathbf{r} &= \int_{\sigma[n]}\left(\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}r^{2k}\right)d\mathbf{r}\\ &= \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\int_{\sigma[n]}r^{2k}d\mathbf{r}\\ &= Vol(\sigma[n]) + \sum_{k=1}^{\infty}\binom{\alpha+k-1}{k}\int_{\sigma[n]}H_{2k}(\mathbf{x},\ldots,\mathbf{x})dx.\end{aligned}$$

Let $A_0 = 1$ and for $k \geq 1$ define the term

$$A_k = \binom{n+2k}{2k}^{-1}\sum_{0\leq i_1\leq i_2,\ldots,\leq i_{2k}\leq n} H_{2k}(v_{i_1},\ldots,v_{i_{2k}})$$

Applying the Lassere-Avrechenkov theorem we transform the integral into the sum

$$Vol(\sigma[n])\left(1+\sum_{k=1}^{\infty}\frac{\binom{\alpha+k-1}{k}}{\binom{n+2k}{2k}}\sum_{0\leq i_1\leq i_2,\ldots,\leq i_{2k}\leq n} H_{2k}(v_{i_1},\ldots,v_{i_{2k}})\right) = Vol(\sigma[n])\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}A_k.$$

We have reduced the comparison of the integrals $\varphi_n(\frac{n+1}{2})$ and $\varphi_n(\frac{n-1}{2})$ over $\sigma[n]$ to a comparison of convergent sums with similar terms.

$$\begin{aligned}\frac{\varphi_n(\alpha+1)}{\varphi_n(\alpha)} &= \frac{\sum_{k=0}^{\infty}\binom{\alpha+k}{k}A_k}{\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}A_k}\\ &= \frac{\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}(1+\frac{k}{\alpha})A_k}{\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}A_k}\\ &= 1+\frac{1}{\alpha}\frac{\sum_{k=0}^{\infty}k\left[\binom{\alpha+k-1}{k}A_k\right]}{\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}A_k}\end{aligned}$$

There are $\binom{n+2k}{2k}$ possible monotonic increasing sequences $0 \leq i_1 \leq \ldots \leq i_{2k} \leq n$ of length $2k$ and $(2k)!$ permutations in the group S_{2k}, thus the term A_k can be seen as the average of all products $\prod_{l=1}^{k}\langle v_{i_{\sigma[n]}}, v_{i_{\sigma[n]}}\rangle$. As k grows, the number of inner products multiplied in each product increases, and $|\langle v_i, v_j\rangle| \leq 1$, so the average of each product summed decreases. Thus we know the quantity A_k decreases, but to estimate this ratio we would need to study the decay rate of the term A_k. For example, if $\binom{\alpha+k-1}{k}A_k$ were to decrease geometrically such that $\binom{\alpha+k-1}{k}A_k = a^k$ for

$|a| < 1$,

$$\frac{\varphi_n(\frac{n+1}{2})}{\varphi_n(\frac{n-1}{2})} = 1 + \frac{2}{n-1}\frac{\sum_{k=0}^{\infty} ka^k}{\sum_{k=0}^{\infty} a^k} = 1 + \frac{2}{n-1}\frac{a}{(1-a)}$$

We can write $H_{2k}(v_1, \ldots, v_{2k})$ in terms of H_{2k-2} for $k \geq 2$. Partition the set of summands of $(2k)!H_{2k}(v_1, \ldots, v_{2k})$ such that each class $[\prod_{l=1}^{k} \langle v_{\pi(2l-1)}, v_{\pi(2l)} \rangle]$ consists of the $2^k k!$ summands with the same factorization into inner products. Fix any $j = 1, \ldots, k$ and any permutation $\pi \in S_{2k}$. There are $2^{k-1}(k-1)!$ permutations of the elements $\{1, \ldots, 2k\} \backslash \{\pi(2j-1), \pi(2j)\}$ such that a summand of $(2k-2)!H_{2k-2}(v_1, \ldots, \widehat{v}_{\pi(2j-1)}, \ldots, \widehat{v}_{\pi(2j)}, \ldots, v_{2k})$ multiplied by $\langle v_{\pi(2j-1)}, v_{\pi(2j)} \rangle$ factors into all of the same factors of $\prod_{l=1}^{k} \langle v_{\pi(2l-1)}, v_{\pi(2l)} \rangle$. As this holds for each $j = 1 \ldots, k$, there are $2^{k-1}k!$ summands in the sum expanded sum of $(2k-2)! \sum_{0 \leq x < y \leq 2k} H_{2k-2}(v_1, \ldots, \widehat{v}_x, \ldots, \widehat{v}_y, \ldots, v_{2k}) \langle v_x, v_y \rangle$ that correspond to $\prod_{l=1}^{k} \langle v_{\pi(2l-1)}, v_{\pi(2l)} \rangle$. This correspondence is unique up to equivalence class and extends to a partition on the sums between these sets so we have

$$\begin{aligned} H_{2k}(v_1, \ldots, v_{2k}) &= \frac{2(2k-2)!}{(2k)!} \sum_{0 \leq x < y \leq 2k} H_{2k-2}(v_1, \ldots, \widehat{v}_x, \ldots, \widehat{v}_y, \ldots, v_{2k}) \langle v_x, v_y \rangle \\ &= \binom{2k}{2}^{-1} \sum_{0 \leq x < y \leq 2k} H_{2k-2}(v_1, \ldots, \widehat{v}_x, \ldots, \widehat{v}_y, \ldots, v_{2k}) \langle v_x, v_y \rangle. \end{aligned}$$

Extending this to A_k,

$$\begin{aligned} A_k &= \binom{n+2k}{2k}^{-1} \binom{2k}{2}^{-1} \sum_{0 \leq i_1 \leq i_2, \ldots, \leq i_{2k} \leq n} \left(\sum_{0 \leq x < y \leq 2k} H_{2k-2}(v_{i_1}, \ldots, \widehat{v}_{i_x}, \ldots, \widehat{v}_{i_y}, \ldots, v_{i_{2k}}) \langle v_{i_x}, v_{i_y} \rangle \right) \\ &= \binom{n+2k}{2k}^{-1} \binom{2k}{2}^{-1} \sum_{0 \leq i_1 \leq i_2, \ldots, \leq i_{2k-2} \leq n} \sum_{0 \leq x < y \leq 2k} \left(H_{2k-2}(v_{i_1}, \ldots, v_{i_{2k-2}}) \sum_{\substack{0, i_{x-1} \leq r \leq i_x, s \\ i_{y-2} \leq s \leq i_{y-1}, n}} \langle v_r, v_s \rangle \right). \end{aligned}$$

References

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(m, n)-pseudoplanes

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Abstract

In this paper, we develop the theory of (m, n)-pseudoplanes, a generalization of projective planes. In particular, we consider the natural analogue of the question posed by Erdős of whether every finite partial plane embeds in a finite projective plane to see that it trivializes in the case of (m, n)-pseudoplanes with m or $n \geq 3$. This fact implies that the theory of generic (m, n)-pseudoplanes has no prime model.

1 Non-degeneracy in (m, n)-pseudoplanes

An incidence structure $(\mathcal{P}, \mathcal{L}, I)$ is a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$, and a binary relation of incidence I between them.

We say a set of points A is incident to a set of lines B when each point in A is incident to each line in B. Abusing notation, we say that a point p is incident to a set of lines B when $\{p\}$ is incident to B, and similarly for a line and a set of points.

Definition 1.1. A (m, n)-*configuration* is an incidence structure such that no set of m points is incident with a set of n lines.

Definition 1.2. When $m, n \geq 2$, a non-degenerate (m, n)-pseudoplane is an incidence structure such that

1. Every set of m points is incident with exactly $n - 1$ lines.
2. Every set of n lines is incident with exactly $m - 1$ lines.
3. (Non-degeneracy) There exist some mn points such that no $m+1$ of them are incident to a single line, and there exist some mn lines such that no $n+1$ of them are incident to a single point.

Projective planes are $(2, 2)$-pseudoplanes. In particular, the existence of a quadrangle in a projective plane implies the existence of its dual structure, a quadrilateral, in that projective plane, and so both halves of the non-degeneracy axiom in Definition 1.2 are satisfied.

The concept of duality in projective planes generalizes to pseudoplanes, but whereas the dual of a projective plane is another projective plane, the dual of an (m, n)-pseudoplane is an (n, m)-pseudoplane: if $(\mathcal{P}, \mathcal{L}, I)$ is an (m, n)-pseudoplane, its dual is $(\mathcal{L}, \mathcal{P}, I^*)$, where $(l, p) \in I^* \iff (p, l) \in I$, an (n, m)-pseudoplane. Statements about (m, n)-pseudoplanes can be dualized by both exchanging "point" for "line"and vice versa and replacing m by n and vice versa. Thus the statement of non-degeneracy in Definition 1.2 consists of the statement, there exist some mn points such that no $m+1$ of them are incident to a single line, and its pseudoplane dual.

The following two propositions are consequences of non-degeneracy; indeed, non-degeneracy will typically be used in one of these forms. It is nevertheless helpful to define non-degeneracy with an existential statement, as above, being both easier to check and more clearly a generalization of non-degeneracy in projective planes.

Proposition 1.3. *In a non-degenerate (m, n)-pseudoplane,*

(a) For each set of $n-1$ lines, there exists a set of $2m-1$ points incident to it.

(b) For each set of $m-1$ points, there exists a set of $2n-1$ lines incident to it.

Proof. Suppose A is a non-degenerate (m, n)-pseudoplane, fix a set S of $n-1$ lines in A, and let T be the set of all points incident to S. We wish to show $|T| \geq 2m-1$. By definition, A contains some set of mn points such that no $m+1$ of them lie on a single line; denote this set of points by Q. No more than m points of Q are incident to any line in S, and so no more than $m(n-1)$ points in Q are incident with any line in S. Thus since $|Q| = mn$, we may pick some $m-1$ points in Q so that no point is incident to any line in S; call this set K.

Any set K of $m-1$ points in Q is incident to at least $(mn-m+1)(n-1)$ lines: for each point $p \in Q \setminus K$, there is a set J_p of $n-1$ lines incident to $\{p\} \cup K$, and the sets J_p are mutually disjoint, since no $m+1$ points in Q are incident to any line.

Thus there is a set R of $(mn-m+1)(n-1)$ lines incident to K. Certainly $R \cap S = \varnothing$, since K is incident to each line in R but to no lines in S. Similarly, $T \cap K = \varnothing$, since the points in T are incident to S but those in K are not.

We now count the number of pairs (p, l) where $p \in T$, $l \in R$, and p is incident to l. On the one hand, for each $l \in R$, there are precisely $m-1$ points incident to $\{l\} \cup S$, which is to say, there are $m-1$ points in T incident to l. Thus there are $|R|(m-1) = (mn-m+1)(n-1)(m-1)$ such pairs. On the other hand, for a point $p \in T$, there are $n-1$ lines incident to $\{p\} \cup K$ and so at most $n-1$ lines in R incident to p. Thus we have the inequality

$$\begin{aligned} |T|(n-1) &\geq (mn-m+1)(n-1)(m-1) \\ |T| &\geq (mn-m+1)(m-1) \end{aligned}$$

Now,

$$\begin{aligned} (mn-m+1)(m-1) &= (m(n-1)+1)(m-1) \\ &= (m(n-1)-(n-1)+1)\,m-1 \\ &= ((m-1)(n-1)+1)m-1 \\ &\geq 2m-1 \end{aligned}$$

since $m \geq 2$, $n \geq 2$. Thus

$$|T| \geq 2m-1$$

as desired. The proof of the dual statement is similar. $\square$

Proposition 1.4. *Suppose A is a non-degenerate (m,n)-pseudoplane. Then for each set S of m points in A, there exist some n lines such that no line is incident with any point in S. Similarly, for each set T of n lines in A, there exist some m points such that no point is incident with any line in T.*

Proof. Suppose S_1 is a set of m points in A. Then there are exactly $n-1$ lines which go through each point in A; call this set T_1. By Proposition 1.3, there are at least $2m-1$ points which are on each line in T_1: all of the points in S_1, of which there are m, and at least $m-1$ other points not in A. Let S_2 be a set of $m-1$ of these points which are not in S_1 but are on each line in T_1. By Proposition 1.3 again, there are at least $2n-1$ lines which go through each point in S_2. T_1 accounts for $n-1$ of these lines, so there are at least n others; Let T_2 be a set n lines not in T_1 which go through each point in S_2.

We claim that no point in S_1 is incident with any line in T_2. Suppose there were some point $p \in S_1$ and line $l \in T_2$ such that p was on l. Then $\{p\} \cup S_2$ is a set of m points each of which lies on each line in $\{l\} \cup T_1$, a set of n lines. But A is an (m,n)-pseudoplane, and so no m points can be incident with n lines. Thus no point in S_1 is incident with any point in T_2, and in general for each set of m points in A, there exists some set of n lines such that none of the m points are incident with any of the n lines. The proof of the dual statement is similar. □

The next proposition generalizes the result in projective planes that if one point has $n+1$ lines incident to it, each point has $n+1$ lines incident to it, and likewise the dual.

Proposition 1.5. *In any non-degenerate (m,n)-pseudoplane with $m, n \geq 2$:*

(a) The number of lines through each set of a points is the same for $1 \leq a \leq m$.

(b) The number of points on each set of b lines is the same for $1 \leq b \leq n$.

Proof. We prove (a) by induction. If $a = m$, the statement is clear, since there are precisely $n-1$ lines which go through each set of m points in any (m,n)-pseudoplane. Thus fix a with $1 \leq a \leq m-1$ and suppose that for each $a < c \leq m$ there is some constant L_c so that there are exactly L_c lines incident with each point in any set of c points. To show (a), then, it suffices to show that, given this induction hypothesis, if A_1 and A_2 are two sets of a points in some non-degenerate (m,n)-pseudoplane such that $|A_1 \cap A_2| = a-1$, then the number of lines which go through each point in A_1 is the same as the number of lines which go through each point A_2. Then for A and A' sets of a points each (with arbitrary intersection) we can construct a chain $A, A_1, \ldots A_{k-1}, A'$ of sets of a points with $k \leq a$ such that two adjacent sets in the sequence share $a-1$ points. By induction and transitivity we can then see that the number of lines which go through each point in A is the same as the number of lines which go through each point A'.

So suppose A_1 and A_2 are two sets of a points in some non-degenerate (m,n)-pseudoplane such that $|A_1 \cap A_2| = a-1$. Then $|A_1 \cup A_2| = a+1 \leq m$, so by Proposition 1.4, there exist some n lines which don't go through any point in A_1 or A_2. Let B be a set of $n-1$ of these lines not incident with any point in $A_1 \cup A_2$. Denote the number of points on each line in B by r_B and the number of lines incident with each point in A_1 by t_{A_1}, and the number of lines incident with each point in A_2 by t_{A_2}.

We count the number of ordered pairs (p, l) where p is a point incident to B, l is a line incident to A_1, and p is incident to l. On the one hand, we can first pick a point p incident to B, for which there are r_B choices, and then pick a line l incident to p and to A. Since

$|A_1| = a$, l is picked incident to a set of $a+1$ points, and so by the induction hypothesis, there are L_{a+1} possibilities for l. Thus the number of ordered pairs (p, l) with the desired properties is $r_B L_{a+1}$. On the other hand, we can first pick the line l incident to A_1, for which there are t_{A_1} choices, and then pick a point p incident to l as well as B, for which there are $m-1$ choices, since $|B \cup \{l\}| = n$, for a total of $t_{A_1}(m-1)$ pairs (p, l) with the desired incidences. Thus $r_B L_{a+1} = t_{A_1}(m-1)$.

An identical argument counting the number of ordered pairs (p, l) where p is a point incident to everything in B, l is a line incident to everything in A_2, and p is incident to l shows $r_B L_{a+1} = t_{A_2}(m-1)$. Thus since $m \geq 2$, $t_{A_1} = t_{A_2}$, as desired.

Thus by the chaining argument and induction, we have showed the number of lines through any set of a points is the same. The proof of (b), the dual statement, is similar. $\square$

Definition 1.6. In a non-degenerate (m, n)-pseudoplane, denote the cardinality of the set of lines incident to each point in a set of a points L_a for $0 \leq a \leq m$. Denote the cardinality of the set of points incident to each line in a set of b lines by P_b for $0 \leq b \leq n$.

Notice that L_0 is just the number of lines and P_0 is just the number of points, and so the constants L_a and P_b are well-defined by Proposition 1.5. In an (m, n)-pseudoplane, it is possible that L_a and P_b are infinite cardinals; the proof of Proposition 1.5 implies the constants are well-defined even when they are infinite.

In a finite projective plane of order n, $P_1 = L_1 = n+1$.

There are tight relationships between the constants L_a and P_b which we will prove in Proposition 1.8 using counting arguments. The following technical lemma ensures the existence of the starting configurations used in these counting arguments.

Lemma 1.7. *In any non-degenerate (m, n)-pseudoplane, if $a, b, r,$ and s are nonnegative integers such that $a \leq m$, $b \leq n$, $r \leq a$, $s \leq b$, if $a = m$ then $s \leq n-1$, if $b = n$ then $r \leq m-1$, and $r = a$ if and only if $s = b$, then there exists a configuration of a set A of a points and a set B of b lines such that there are exactly r points in A incident with everything in B, and exactly s lines in B incident with everything in A.*

Proof. In a non-degenerate (m, n)-pseudoplane, there are at least m points, and by Proposition 1.4 there are n lines with no incidences to any of the m points. Thus since $a - r \leq m$ and $b - s \leq n$, certainly we can find a set A_1 of $a - r$ points and and a set B_1 of $b - s$ lines with no incidences between any of these points and lines. Now, if $a - r < m$, there are at least $2n-1$ lines incident to A_1, and so we can choose s of these lines to form a set B_2. On the other hand, if $a - r = m$, then since $a \leq m$ and $0 \leq r \leq a$, we have $a = m$ and $r = 0$. Then $s \leq n-1$ is given. The number of lines through all $a - r$ points is $n-1$, and so since $s \leq n-1$ we may pick s lines incident with everything in A_1 to form a set B_2.

Notice $B_1 \cap B_2 = \varnothing$ if $a \neq r$, since the lines in B_1 are not incident to the points in A_1 but the lines in B_2 are. Thus $|B_1 \cup B_2| = (b-s) + s = b$. Indeed, even if $a = r$, $b = s$ is given; then $B_1 = \varnothing$, $B_1 \cup B_2 = B_2$, and $|B_1 \cup B_2| = s = b$. So regardless, we see $|B_1 \cup B_2| = b$. Then if $b < n$, there are at least $2m - 1 \geq m$ points on all lines in B_2 and so certainly we may choose a set A_2 of $r \leq m$ points incident to everything in B_2. On the other hand, if $b = n$, then $r \leq m-1$ is given, and again we can choose A_2 to contain r points so that each point is incident to B_2.

Now $A_1 \cap A_2 = \varnothing$ if $b - s \neq 0$, since A_2 is incident to those things in B_1 but A_1 is not; and even if $b - s = 0$, then also $a - r = 0$ is given, so A_1 is empty anyways. Either way, we see $|A_1 \cup A_2| = a$.

Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. A is a set of a points and B is a set of b lines. The points in A incident with everything in B are precisely the points in A_2, of which there are r. The lines in B incident with everything in A are precisely the lines in B_2, of which there are s. Thus we have shown the existence of the desired configuration. $\square$

When κ is an infinite cardinal and a is a nonnegative integer, we denote by $\kappa - a$ the cardinality of a set of κ elements from which a elements have been removed, and we denote by $\binom{\kappa}{a}$ the cardinality of the set of subsets of size a of a set of size κ. Note that $\kappa - a = \binom{\kappa}{a} = \kappa$ if $a \geq 1$.

Proposition 1.8. *In any non-degenerate (m, n)-pseudoplane, if a, b, c, d, r, and s are non-negative integers such that $a + c \leq m$, $b + d \leq n$, $r \leq a$, $s \leq b$, if $a = m$ then $s \leq n - 1$, if $b = n$ then $r \leq m - 1$, and $r = a$ if and only if $s = b$, then:*

$$\binom{L_a - s}{d}\binom{P_{b+d} - r}{c} = \binom{P_b - r}{c}\binom{L_{a+c} - s}{d} \tag{1}$$

Proof. By Lemma 1.7 we can find a configuration of a set A of a points and a set B of b lines such that there are exactly r points in A incident with everything in B, and exactly s lines in B incident with everything in A. We count the cardinality of the set S of ordered pairs of sets (C, D) where C is a set of c points, D is a set of d lines, $C \cap A = \varnothing$, $D \cap B = \varnothing$, everything in A is incident with everything in D, everything in C is incident with everything in B, and everything in C is incident with everything in D. If we choose the set D of d lines first, we need D incident to everything in B and $D \cap B = \varnothing$; there are L_a lines incident to everything in A, but exactly s of those lines are in B. Thus there are $\binom{L_a-s}{d}$ choices for D, if we choose it first. Then we pick a set C of c points such that everything in C is incident with everything in B and D (of which there are P_{b+d} points) and is disjoint from A (which has r points incident to everything in $B \cup D$), for $\binom{P_{b+d}-r}{c}$ choices of C given D. Thus $|S| = \binom{L_a-s}{d}\binom{P_{b+d}-r}{c}$. On the other hand, we can count the choices for C first, and multiply by the choices for D given C: in this case, there are $\binom{P_b-r}{c}$ choices for a set C incident with everything in B and disjoint from A and $\binom{L_{a+c}-s}{d}$ choices for a set D incident with everything in $A \cup C$ and disjoint from B. Thus also $|S| = \binom{P_b-r}{c}\binom{L_{a+c}-s}{d}$, and so (1) holds, as desired. $\square$

The identities in Proposition 1.8 are quite constraining; indeed, as we see in the next theorem, in finite (m, n)-pseudoplanes which are not projective planes, the identities are inconsistent, allowing us to conclude there are no finite (m, n)-pseudoplanes with $m \geq 3$ or $n \geq 3$.

Theorem 1.9. *All non-degenerate (m, n)-pseudoplanes with $m \geq 3$ or $n \geq 3$ are infinite.*

Proof. Suppose there were some finite non-degenerate (m, n)-pseudoplane A for $m \geq 3$ and $n \geq 2$. We substitute $a = m - 1, b = 1, c = d = 1$, $r = 1$, $s = 0$ in 1 (since they clearly satisfy the hypotheses) to see

$$L_{m-1}(P_2 - 1) = (P_1 - 1)L_m.$$

Furthermore, with $a = m - 1, b = 1, c = d = 1$, $r = s = 0$, we see from 1,

$$L_{m-1}P_2 = P_1 L_m.$$

Since A is finite, all of the constants L_a and P_b are constant; thus, subtracting, we see

$$L_{m-1} = L_m = n - 1.$$

But by Proposition 1.3, $L_{m-1} \geq 2n - 1 > n - 1$, a contradiction. Thus there can be no finite non-degenerate $(m, n)-$pseudoplanes for $m \geq 3$ and $n \geq 2$.

The proof of the case $m \geq 2$, $n \geq 3$ is similar, with substituting $a = 1, b = n-1, c = d = 1$, $r = 0$, $s = 1$ in (1) to see $(L_1 - 1)P_n = P_{n-1}(L_2 - 1)$ and $a = 1, b = n-1, c = d = 1$, $r = s = 0$ in (1) to see $L_1 P_n = P_{n-1} L_2$. Then $P_{n-1} = P_n$, contradicting Proposition 1.3. Thus the only finite non-degenerate (m, n)-pseudoplanes are projective planes, as desired. □

The following questions about non-degeneracy in (m,n)-pseudoplanes remain open:

1. There is a well-known classification of the degenerate projective planes. Can the non-degenerate (m, n)-pseudoplanes be similarly classified?
2. Does the condition that there are mn points, no $m+1$ of which are incident to a single line imply its dual?
3. Can our non-degeneracy condition be replaced with a simpler (and self-dual) condition which still suffices for the results in this section?

2 The existence and properties of (m, n)-pseudoplanes

While there are no finite non-degenerate (m, n)-pseudoplanes, there are plenty of infinite ones.

Definition 2.1. Suppose $A = (\mathcal{P}_0, \mathcal{L}_0, I_0)$ is an (m, n)-configuration. Define $F_0(A) = A$. Then for each $k \geq 0$, given an (m, n)-configuration $F_k(A) = (\mathcal{P}_k, \mathcal{L}_k, I_k)$, define the (m, n)-configuration $F_{k+1}(A) = (\mathcal{P}_{k+1}, \mathcal{L}_{k+1}, I_{k+1})$ as follows:

1. $\mathcal{P}_{k+1}$ includes all the points in $\mathcal{P}_k$. Furthermore, for each set S_i of n lines in $\mathcal{L}_k$, let d_i be the number of points incident to S_i. Then $\mathcal{P}_{k+1}$ also contains $m - 1 - d_i$ new, distinct points incident to S_i.
2. $\mathcal{L}_{k+1}$ includes all the lines in $\mathcal{L}_k$. Furthermore, for each set T_j of m points in $\mathcal{P}_k$, let e_j be the number of points incident to T_j. Then $\mathcal{L}_{k+1}$ also contains $n - 1 - e_j$ new, distinct lines incident to T_j.
3. I_{k+1} is I_k, plus the incidences involving the added points and lines described above.

Note that, by induction, each $F_k(A)$ is in fact an (m, n)-configuration, as in Conant and Kruckman [1, Proposition 2.3].

Let $\mathcal{P} = \cup_{k=0}^{\infty} \mathcal{P}_k$, $\mathcal{L} = \cup_{k=0}^{\infty} \mathcal{L}_k$, $I = \cup_{k=0}^{\infty} I_k$. Then the *free completion* of the configuration $A_0 = (\mathcal{P}_0, \mathcal{L}_0, I_0)$ is $F(A) = (\mathcal{P}, \mathcal{L}, I)$

Proposition 2.2. *The free completion of an (m, n)-configuration is a (possibly degenerate) (m, n)-pseudoplane. The free completion of an (m, n)-configuration which contains some mn points such that no $m+1$ are incident to a single line and contains some mn lines such that no $n+1$ are incident to a single point is a non-degenerate (m, n)-pseudoplane.*

Proof. The first statement is Proposition 2.3 in [1, p. 3]. The second is by the definition of non-degeneracy, since A embeds in $F(A)$ and so the configurations contained in A are also contained in $F(A)$. $\square$

In infinite (m, n)-pseudoplanes, the constants L_a and P_b are well-defined as infinite cardinals for which the combinatorial relationships found in Section 2 still stand.

Lemma 2.3. *In an infinite non-degenerate (m, n)-pseudoplane, the constants L_a and P_b are the same infinite cardinal, for $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$*

Proof. In an infinite (m, n)-pseudoplane, some L_i or P_j will be infinite. Since $L_0 \geq L_i$ and $P_0 \geq P_j$, either P_0 or L_0 will be infinite also. Then $P_0 = L_0$:

Setting $a = r = 0, b = s = 0, c = m-1, d = n$ in (1) yields

$$\binom{L_0}{n} = \binom{P_0}{m-1}\binom{L_{m-1}}{n} \tag{2}$$

and similarly $a = r = 0, b = s = 0, c = m, d = n-1$ in (1) yields

$$\binom{L_0}{n-1}\binom{P_{n-1}}{m} = \binom{P_0}{m}. \tag{3}$$

Thus if L_0 is infinite, equation (3) shows P_0 must be infinite also, and in fact $P_0 \geq L_0$. Then equation (2) yields $L_0 \geq P_0$, and so together we have $P_0 = L_0$. We get the same result starting with P_0 infinite and using equation (2) and then (3).

Now, using $a = r = 0$, $b = s = n-1$, $c = m-1$, $d = 1$ in (1) yields

$$(L_0 - n + 1)\binom{m-1}{m-1} = \binom{P_{n-1}}{m-1}(L_{m-1} - n + 1) \tag{4}$$

Thus, if L_0 is an infinite cardinal, both sides of the equation will be infinite, and so at least one of P_{n-1} and L_{m-1} will be infinite also.

Using $r = a-1, s = b-1, c = m-a, d = n-b$ in (1) yields that for all $1 \leq a \leq m$ and $1 \leq b \leq n$,

$$\binom{L_a - b + 1}{n-b}\binom{m-a}{m-a} = \binom{P_b - a + 1}{m-a}\binom{n-b}{n-b}$$
$$\binom{L_a - b + 1}{n-b} = \binom{P_b - a + 1}{m-a}. \tag{5}$$

At $a = m-1$, equation (5) holds for all $1 \leq b \leq n$ as

$$\binom{L_{m-1} - b + 1}{n-b} = (P_b - m + 2) \tag{6}$$

So if L_{m-1} is infinite, for $1 \leq b \leq n-1$, P_b is infinite also. Likewise, at $b = n-1$ equation (5) holds for all $1 \leq a \leq m$ as

$$(L_a - n + 2) = \binom{P_{n-1} - a + 1}{m-a} \tag{7}$$

So if P_{n-1} is infinite, for $1 \leq a \leq m-1$, L_a is infinite also.

Thus, if L_0 is infinite, equation (4) shows that at least one of L_{m-1} and P_{n-1} is also infinite. In the former case, equation (6) yields that for $1 \leq b \leq n-1$, P_b is infinite also, and in fact is equal to L_{m-1}. Then, in particular, P_{n-1} is infinite and equal to L_{m-1}, so equation (7) yields that L_a is infinite and equal to L_{m-1} for all $1 \leq a \leq m-1$ also. Then (4) shows that $L_0 = P_{n-1}L_{m-1} = L_{m-1}^2 = L_{m-1}$. Since $L_0 = P_0$, we have that the constants L_a and P_b are the same infinite cardinal, for $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$.

If instead we take (4) with P_{n-1} infinite, the same argument shows the same result, but with (6) and (7) used in the opposite order. □

Corollary 2.4. *In an infinite (m,n)-pseudoplane, P_0 and L_{m-1} are infinite.*

Proof. In an infinite (m,n)-pseudoplane, some constant L_a or P_b is infinite, with $0 \leq a \leq m-1$ or $0 \leq b \leq n-1$. Then P_0 and L_{m-1} are both equal to that infinite cardinal, by Lemma 2.3. □

3 Model theoretic consequences

We consider (m,n)-configurations in the language $\mathcal{L} = \{P, L, I\}$, where P is a unary predicate for points, L is a unary predicate for lines, and I is a binary relation for incidence between points and lines.

The theory of (m,n)-pseudoplanes has a model companion, $T_{m,n}$, as in [1, p. 4]. Models of $T_{m,n}$ are the existentially closed (m,n)-pseudoplanes.

The following background is found in [1].

Definition 3.1. Let A be an (m,n)-configuration with elements labeled (possibly with repeats) by the variables $\bar{x}$. The *diagram* $\mathrm{Diag}_A(\bar{x})$ is the set of all atomic and negated atomic formulas true in A.

As in [1, p. 4], when A is finite, we identify $\mathrm{Diag}_A(\bar{x})$ with the formula $\bigwedge_{\varphi \in \mathrm{Diag}_A(\bar{x})} \varphi$.

Definition 3.2. Fix an (m,n)-pseudoplane B. We say that a subset $A \subseteq B$ is *I-closed (in B)* if, for all pairwise distinct points $a_1, \ldots, a_m \in A$, if some line $b \in B$ is incident to each of the a_i, then $b \in A$, and, dually, for all pairwise distinct lines $b_1, \ldots, b_n \in A$, if some point $a \in B$ is incident to each of the b_j, then $a \in A$. The *I-closure in* B of a set $A \subseteq B$ is the smallest I-closed subset of B containing A. If the I-closure of A in B is all of B, then we say A *generates* B.

Proposition 3.3 ([1, Corollary 2.15])**.** *Two tuples a and a' have the same type if and only if there is an isomorphism of their closures which sends a to a'.*

Definition 3.4. Given an (m,n)-configuration A, a *basic existential formula* is one of the form

$$\exists \bar{y}\ \mathrm{Diag}_A(\bar{x}, \bar{y})$$

where $\mathrm{Diag}_A(\bar{x}, \bar{y})$ implies that $\bar{x}$ generates $\overline{xy}$.

While $T_{m,n}$ does not have full quantifier elimination, it does have almost quantifier elimination.

Proposition 3.5 ([1, Proposition 2.17])**.** *Modulo $T_{m,n}$, every formula is equivalent to a finite disjunction of basic existential formulas.*

Lemma 3.6. *Suppose A is a finite partial $(m, n)-$pseudoplane whose free completion $F(A)$ is infinite and non-degenerate. Then there is an (m, n)-pseudoplane B such that A embeds in B and generates B, but B and $F(A)$ are not isomorphic over A.*

Proof. We inductively find lines $r_1, \ldots, r_{nm+2n+2}$ so that no $n+1$ lines are incident with a single point, and if k_i is the minimal such that $r_i \in F_{k_i}(A)$, then $k_1 < \cdots < k_{nm+2n+2}$ (i.e., the line r_j first appears in a strictly later step of the free completion than all the r_i for $i < j$). Let r_1 be any line in $F(A)$. Now suppose we have found r_i for $i \le l < nm + 2n + 2$. Let X be the set of all intersection points of some n of the lines r_i for $i \le l$ in $F(A)$. The set X is finite, since n lines intersect at exactly $m-1$ points in the free completion, so certainly we can pick $m-1$ points $d_1, \ldots d_{m-1}$ in $F(A) \setminus X$, since $F(A)$ is infinite and so has infinitely many points by Corollary 2.4. There are infinitely many lines through $d_1, \ldots d_{m-1}$, again by Corollary 2.4, and so we can choose one which is not in $F_{k_l}(A)$ and not incident with any point in X to serve as r_{l+1}; $F_{k_l}(A)$ is finite, X is finite, and for each point $x \in X$, there are only $n-1$ lines through $d_1, \ldots, d_{m-1}$, and x.

Partition the $nm+2n+2$ lines into four sets (of cardinalities $m+1$, $n+1$, $(m+1)(n-1)$, and 1, respectively) as follows:

$$
\begin{aligned}
S_1 &= \{r_i : 1 \le i \le m+1\} \\
S_2 &= \{r_i : (m+1)+1 \le i \le (m+1)+(n+1)\} \\
S_3 &= \{r_i : (m+1)+(n+1)+1 \le i \le (m+1)+(n+1)+(m+1)(n-1)\} \\
S_4 &= \{r_i : i = (m+1)+(n+1)+(m+1)(n-1)+1\}.
\end{aligned}
$$

Let $k = k_{nm+2n+2}$. In $F_k(A)$, $r_{nm+2n+2}$ is incident with m points, each of which lies on at most $n-1$ lines r_i in $S_1 \cup S_2 \cup S_3$, by construction. So since $|S_3| = (m+1)(n-1)$, there are at least $n-1$ lines in S_3 without any intersection points to $r_{nm+2n+2}$ in $F_k(A)$. Thus, in $F_{k+1}(A)$ there are $m-1$ new points $b_1, \ldots, b_m - 1$ lying on $r_{nm+2n+2}$ and some $n-1$ points in S_3.

There is no line in $F_{k+1}(A)$ through $b_1, \ldots, b_{m-1}$ and any intersection point of any n lines in S_2, since the b_i in $F_{k+1}(A)$ lie only on n lines in S_3 and S_4. Thus in $F_{k+2}(A)$ there are distinct connecting lines through $b_1, \ldots, b_{m-1}$ and each intersection point of n lines in S_2. There are $n+1$ lines in S_2, and so there are $n+1$ collections of n lines in S_2, each of which has $m-1$ intersection points; and each of those intersection points is being paired with $b_1, \ldots b_{m-1}$ to yield $n-1$ lines which go through all of them; and so this construction yields $(n+1)(m-1)(n-1)$ new distinct lines in $F_{k+2}(A)$. Call this set of lines C.

Since $n(m-1) \ge 2$, we have $n(m-1) + m - 1 \ge m + 1$, and $(n+1)(m-1) \ge (m+1)$. Thus $|C| = (n+1)(m-1)(n-1) \ge (m+1)(n-1)$. Therefore we can construct $m+1$ mutually disjoint sets each containing $n-1$ of the lines in C as well as one line in S_1. Since no line in S_1 is incident to any b_i or intersection point of n lines in S_2 by construction, each of these new sets has no intersection points. Thus in $F_{k+3}(A)$ each will give rise to $m-1$ intersection points, and in particular there will be some $c_1, \ldots c_{m+1}$ not on the same line in $F(A)$.

However, we can extend $F_{k+3}(A)$ to a larger partial (m, n)-pseudoplane B_0 by adding a line incident to each of $c_1, \ldots c_{m+1}$. Then A embeds in $F(B_0)$ and generates $F(B_0)$, but $F(B_0)$ and $F(A)$ are not isomorphic over A. □

We will use the following variant of Lemma 3.6 in the proof of Theorem 3.7 below:

Suppose A is a finite (m,n)-configuration, such that any completion of A is non-degenerate. Then for any (m,n)-configuration B containing A and generated by A, B has multiple completions which are not isomorphic over A.

A final version of this paper will contain a proof of the above statement.

Recall that a model M of a theory T is a *prime model* if for all models N of T, there is an elementary embedding of M into N. A countable complete theory with infinite models has a prime model if and only if every formula is contained in a complete type which is isolated by a single formula [3, p. 168].

Theorem 3.7. *$T_{m,n}$ does not have a prime model if $m \geq 3$ or $n \geq 3$.*

Proof. Let A be a finite (m,n)-configuration which contains some mn points such that no $m+1$ are incident to a single line and contains some mn lines such that no $n+1$ are incident to a single point. Let $\bar{x}$ enumerate A. We claim that $\mathrm{Diag}_A(\bar{x})$ is not contained in a complete, isolated type.

Suppose for contradiction $p(\bar{x})$ is a complete isolated type which contains $\mathrm{Diag}_A(\bar{x})$. By almost quantifier elimination, there is some formula $\varphi(\bar{x}) = \bigvee_{i=1}^{k} \theta_i(\bar{x})$ where the formulas $\theta_i(\bar{x})$ are basic existential formulas, which isolates $p(\bar{x})$ 3.5. Then there is some $\theta_i(\bar{x})$ such that $\theta_i(\bar{x}) \in p(x)$. Certainly $\theta_i(\bar{x})$ implies $\bigvee_{i=1}^{k} \theta_i(\bar{x})$, which in turn implies the type p, and so $\theta_i(\bar{x})$ itself isolates $p(\bar{x})$. Thus a basic existential formula $\theta_i(\bar{x}) = \exists \bar{y}\ \mathrm{Diag}_B(\bar{x},\bar{y})$, where B is an (m,n)-configuration into which A embeds, isolates the complete type p.

B, since it contains A, contains some mn points such that no $m+1$ are incident to a single line and contains some mn lines such that no $n+1$ are incident to a single point. Thus the free completion of B is infinite: $F(B)$ is an non-degenerate (m,n)-pseudoplane, by Proposition 2.2, and all non-degenerate (m,n)-pseudoplanes are infinite, by Theorem 1.9. But then by Lemma 3.6, there are multiple completions of B (and thus also of A) which are not isomorphic over A. But then $\theta_i(\bar{x}) = \exists \bar{y}\ \mathrm{Diag}_B(\bar{x},\bar{y})$ cannot have isolated a complete type, by 3.3. Thus $T_{m,n}$ cannot not have a prime model, as desired. $\square$

References

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Homotopy Group Calculations in Homotopy Type Theory

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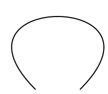
1 Abstract

Homotopy Type Theory is an alternative foundational system for mathematics which provides a synthetic approach to homotopy theory. In this paper, we use homotopy type theory to demonstrate an invariant which maps $\Omega^3(S^2)$ to $\mathbb{Z}$. We then show that $\mathbb{Z} \leq \pi^3(S^2)$ by proving that the invariant maps the hopf construction to 1.

2 Introduction

The framework of Homotopy Type Theory allows us to examine properties of topological spaces from an abstracted perspective. Types correspond to spaces, and the elements of the types correspond to the points in the spaces. Furthermore, a proof that two elements of a type are equal corresponds to a path in the corresponding space. For example, the circle S^1 corresponds to a type generated by one element $base_1$, and one nontrivial way to prove that that element is equal to itself, visualized with a diagram like this:



$base_1$

The type of all loops around $base_1$, or proofs that $base_1$ equals itself, is written $base_1 = base_1$. When a type only has one point (or is connected), we often want an abbreviated notation for loops around that point. We use the operator Ω to mean loop space, so for example $\Omega(S^1) \equiv (base_1 = base_1)$. Similarly, we use Ω^n for the nth iterated loop space. For example if type T has only one point t, then $\Omega^2(T) \equiv (\text{refl}_t = \text{refl}_t)$

Central to calculating homotopy groups of types are computations with path spaces of types. However, in general, computing any information about path spaces is difficult. Even the quesiton of whether two representations of a path are equal is difficult. We solve this problem by mapping the path space of a type onto part of the path space of the universe $\mathcal{U}$. This allows us to take advantage of univalence, which tells us that paths are no more than equivalences. Furthermore, 2-paths in $\mathcal{U}$ turn out to be no more that homotopies between equivalences, and 3-paths and beyond correspond to homotopies between homotopies. This correspondence is useful because both equivalences and homotopies are special types of functions, and one can extract information from functions by plugging in inputs to them.

3 Equivalences

We now explicitly write down the correspondence between paths in $\mathcal{U}$ and equivalences/homotopies. Firstly, 1-paths in $\mathcal{U}$ may be represented by equivalences as univalence directly tells us

$$(A = B) = (A \simeq B)$$

The correspondence between higher paths and homotopies can be written down as follows for loops

$$\Omega^{n+1}(\mathcal{U}, T) = \prod_{t:T} \Omega^n(T, t)$$

We need to prove this and construct explicit equivalences between these two types for n up to 3. It turns out that the right side of the above equality can be written as a homotopy of two functions. To make use of this, we define for any $A : \mathcal{U}$ three functions

$$\begin{array}{l|l|l} \mathrm{id}_1 : A \to A & \mathrm{id}_2 : \mathrm{id}_1 \sim \mathrm{id}_1 \equiv \prod_{a:A} a = a & \mathrm{id}_3 : \mathrm{id}_2 \sim \mathrm{id}_2 \equiv \prod_{a:A} \mathrm{refl}_a = \mathrm{refl}_a \\ \mathrm{id}_1\ a = a & \mathrm{id}_2\ a = \mathrm{refl}_a & \mathrm{id}_3\ a = \mathrm{refl}_{\mathrm{refl}_a} \end{array}$$

We can now rephrase the above type equality and prove it. We define a type equivalence

$$\mathrm{eq}_n : \Omega^{n+1}(\mathcal{U}, T) \to \mathrm{id}_n \sim \mathrm{id}_n$$

With

$$\begin{aligned} \mathrm{eq}_2 &\equiv \mathrm{ap}_{\mathrm{funext}} \circ \mathrm{idtoequiv} \\ \mathrm{eq}_3 &\equiv \mathrm{ap}_{\mathrm{ap}_{\mathrm{funext}}} \circ \mathrm{ap}_{\mathrm{funext}} \circ \mathrm{idtoequiv} \end{aligned}$$

For $n = 2, 3$. We also define for $n = 1$

$$\mathrm{eq}_1 \equiv \mathrm{idtoequiv}$$

Which parallels the above definitions but does not have the same type, because there is no id_0.

4 Invariant

We would like to distinguish between 3-loops in S^2. To do so, we will define a map which maps the entire path space of S^2 into part of the path space of $\mathcal{U}$. Specifically, we will map to the path space around the type $S^1 \times S^2$

$$\begin{aligned}
&\mathrm{code} : S^2 \to \mathbb{U} \\
&\mathrm{code}\ \mathrm{base}_2 \equiv S^1 \times S^2 \\
&\mathrm{ap}^2_{\mathrm{code}}\ \mathrm{surf} = \mathrm{eq}_2^{-1}(\text{two-loop})
\end{aligned}$$

In the last piece, we must provide an element of $\Omega^2(\mathcal{U}, S^1 \times S^2)$. To do so, we use the type equivalence eq_2 defined earlier, and supply an element two-loop : $\prod_{t:S^1 \times S^2} t = t$ The definition of two-loop requires the induction principle for both circles and spheres. The induction principles end up requiring various n-loops in S^1 and S^2. In these locations, we put *loop* or *surf* when possible, and otherwise refl.

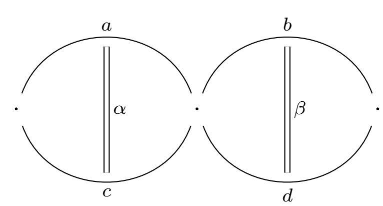
Using *code*, it is easy to define the invariant itself.

$$\begin{aligned}
&\mathrm{invariant} : \Omega^3(S^2) \to \Omega^2(S^2) \\
&\mathrm{invariant}\ l = \mathrm{eq}_3^{-1}(\mathrm{ap}^3_{\mathrm{code}} l)(\mathrm{base}_1, \mathrm{base}_2)
\end{aligned}$$

The hard part will be to prove that the invariant of the hopf construction actually gives something nontrivial.

5 Hopf Construction

In order to define the hopf construction, we will need to define horizontal composition. In a type A, given points and paths as in the diagram below,



We define

$$\alpha * \beta : a \circ b = c \circ d$$

By path induction on α and then β. We also define a second kind of horizontal composition $\alpha *' \beta$, where now we induct first on β and second α.

Next, we define a function hopf, which takes a 2-loop and gives a 3-loop. Given a 2-loop p around a point a, it constructs a three loop by composing five paths through the following types

$$\text{refl}_{\text{refl}_a} = p^{-1} \cdot p = p *' p^{-1} = p * p^{-1} = p \cdot p^{-1} = \text{refl}_{\text{refl}_a}$$

The hopf function will create these five paths with five different functions, and compose the paths together

$$\begin{aligned}
&\text{hopf} : \Omega^2(A) \to \Omega^3(A) \\
&\text{hopf } p = \text{ri}(p) \cdot \text{same'}(p, p^{-1}) \cdot \text{agree}(p, p^{-1}) \cdot \text{same}(p, p^{-1}) \cdot \text{li}(p)
\end{aligned}$$

Next, we will define all five of these functions. We define both of *ri* and *li* (right inverse law and left inverse law) on all paths p

$$\begin{aligned}
&\text{ri}(p) : \text{refl}_{\text{refl}_a} = p^{-1} \cdot p \\
&\text{li}(p) : p \cdot p^{-1} = \text{refl}_{\text{refl}_a}
\end{aligned}$$

by path induction on p. Next, we need to define the functions *same* and *same'*, with types

$$\begin{aligned}
&\text{same} : \prod_{p:\Omega^2(A)} p * p^{-1} = p \cdot p^{-1} \\
&\text{same'} : \prod_{p:\Omega^2(A)} p^{-1} \cdot p = p *' p^{-1}
\end{aligned}$$

It turns out that these two functions don't even need path induction, as horizontal and vertical composition turn out to be judgementally equal on loops. Both same and same' are just defined as refl.

Finally, we define *agree*

$$\text{agree} : \prod_{p:\Omega^2(A)} p *' p^{-1} = p * p^{-1}$$

By path induction on p.

Now, finally, we can try to compute *invariant(hopf(surf))*, but simply plugging in the definitions of *hopf* and *invariant* yields no obvious way to proceed with the computation. We will need to make use of the correspondence between paths in $\mathcal{U}$ and equivalences/homotopies from section 3. Therefore, we define a function

$$
\begin{aligned}
&\mathrm{hopf}_h : \mathrm{id}_1 \sim \mathrm{id}_1 \to \mathrm{id}_2 \sim \mathrm{id}_2 \\
&\quad \equiv \prod_{a:A} a = a \to \prod_{a:A} \mathrm{refl}_a = \mathrm{refl}_a \\
&\mathrm{hopf}_h p = \mathrm{ri}_h(p) \cdot \mathrm{same'}_h(p, p^{-1}) \cdot \mathrm{agree}_h(p, p^{-1}) \cdot \mathrm{same}_h(p, p^{-1}) \cdot \mathrm{li}_h(p)
\end{aligned}
$$

Which produces homotopies through the following types:

$$\mathrm{id}_1 \sim p^{-1} \cdot p \sim p *' p^{-1} \sim p * p^{-1} \sim p \cdot p^{-1} \sim \mathrm{id}_1$$

In an analagous way to *hopf*. Now we must define the "h" (stands for homotopy) version of each of the five functions, as well as define horizontal composition on homotopies. Given

$$
\begin{tikzcd}
A \arrow[r, bend left=60, "f_1"] \arrow[r, bend right=60, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left=60, "g_1"] \arrow[r, bend right=60, "g_2"'] \arrow[r, phantom, "\Downarrow\beta"] & C
\end{tikzcd}
$$

Where A, B, and C are types, the arrows are equivalences, and the squiggly arrows are homotopies between equivalences. We define the two kinds of horizontal composition

$$
\begin{aligned}
&\alpha * \beta, \alpha *' \beta : a \circ b \sim c \circ d \\
&\alpha * \beta(a) \equiv \mathrm{ap}_{g_1} \alpha(a) \cdot \beta(f_2(a)) \\
&\alpha *' \beta(a) \equiv \beta(f_1(a)) \cdot \mathrm{ap}_{g_2} \alpha(a)
\end{aligned}
$$

Now, we can define the five functions used in *hopf*$_h$. First, we define ti_h and li_h in terms of the corresponding operations on paths.

$$
\begin{aligned}
&\mathrm{ri}_h(p, a) \equiv \mathrm{ri}(P(a)) \\
&\mathrm{li}_h(p, a) \equiv \mathrm{li}(P(a))
\end{aligned}
$$

Note that the above functions are defined on any homotopy. By contrast, we define same_h and $\mathrm{same'}_h$ only on loops. Just like the corresponding operations on paths can simply be defined with refl, as horizontal and vertical composition on loops turns out to be judgementally the same

$$\text{same}_h(a) \equiv \text{refl}$$
$$\text{same'}_h(a) \equiv \text{refl}$$

Finally, we define *agree*$_h$ on any two homotopies α and β which can be horizontally composed, as in the diagram from before. It has the type

$$\begin{aligned} &\text{agree}(\alpha, \beta) : \alpha * \beta \sim \alpha *' \beta \\ \equiv \quad &\prod_{a:A} \text{ap}_{g_1}\alpha(a) \cdot \beta(f_2(a)) \equiv \beta(f_1(a)) \cdot \text{ap}_{g_2}\alpha(a) \end{aligned}$$

How can we get something of such a type? No doubt we could do so with path induction, but that would defeat the entire point of looking at paths as homotopies. The entire advantage of doing so is that homotopies are functions. We would like to make use of the expression

$$\text{apd}_\beta \alpha(a)$$

Because using this will be necessary in order for the computation to work. However, exactly how to do this will be a subject of further research.

So we know that *hopf*$_h$ can create a nontrivial 3-loop in homotopy form, but in order to conclude anything about S^2, we need to prove that *hopf* and *hopf*$_h$ in some sense agree with each other. That is, we will prove the following

$$\text{hopf}_h = \text{eq}_3^{-1} \circ \text{hopf} \circ \text{eq}_2$$

But remember that *hopf* and *hopf*$_h$ are both made up of five corresponding functions. We will prove that each of the corresponding functions matches up. That is for $f =$ *ri, same', agree, same,* or *li*,

$$f = \text{eq}_3 \circ f_h \circ \text{eq}_2^{-1}$$

This proof is just an application of path induction on all paths for *li, ri,* and *agree*. For *same and same'*, the two sides of the equation are judgementally equal.

6 Conclusion

We have shown how to define the hopf construction of a 3-loop given a 2-loop, and given a way to calculate it. When *surf* is plugged in, the method above shows that the result is nontrivial. Because of the functorality of functions, we know that powers of the 3-loop are also nontrivial and all different, so we know that a copy of $\mathbb{Z}$ is contained in the 3-loop space of S^2. In the future, we would like to extend this to a proof that $\pi_3(S^2) = \mathbb{Z}$, and also define a similar invariant on 4-loops in S^3. In general, we believe that this method will allow one to compute the loop space of any type, given that they are able to guess in advance what it should look like, perhaps using intuition from classical algebraic topology.

7 Acknowledgements

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MIXING TIMES OF RANDOM WALKS ON VARIOUS COMBINATORIAL OBJECTS

JAMES MARSHALL REBER

Abstract. A classic mixing time result, due to Bayer and Diaconis, is on how many riffle shuffles are required to shuffle a deck of n cards - the required number is $\frac{3}{2}\log_2(n)$. In this paper, we explore the mixing times of random walks on various graphs using a combinatorial method called coupling. In particular, we give upper bounds on the mixing times of certain kinds of random walks on diagonal gluings of 2-regular graphs, repeated gluings of copies of the complete graph of size three, certain families of 3-regular graphs, and conjecture some possible generalizations.

Contents

1. Introduction

When describing the mixing time of random walks, we are really discussing what it means to be "sufficiently close to random" after a certain amount of time, or rather how long would we need to run a Markov chain until it is within $\epsilon > 0$ of its stationary distribution. The prototypical example of this comes from shuffling a deck. When shuffling a deck, we would like to guarantee that we have a fair deck, or one in which all possible ordering of cards are possible. The mixing times of various shuffles, such as the random transposition shuffle, the riffle shuffle, and top-to-random shuffle, have all been studied and have tight bounds on their mixing times [1]. Mixing times can, however, be studied in contexts other than card shufflings; for example, one can ask how long it takes a random walker on a graph to be "sufficiently random" among all possible states on the graph.

Throughout this paper, we are concerned with finding the mixing times of random walks on some special classes of graphs. In Section 1.1 and 1.2, we outline some of the necessary theory and notation from Markov chains and graph theory. In Section 2, we will explore the mixing times of random walks on 2-regular graphs glued diagonally (see Figures 3 and 4 respectively). In Section 3, we will explore the mixing times of random walks on a graph formed by gluing copies of triangles

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(or the complete graph on three vertices) on each vertex of degree two repeated k times (see Figure 10), as well as discuss a result on the mixing time of a random walk on the complete graph glued repeatedly along a single vertex (see Figures 6 and 7). Finally, in Section 4, we will explore mixing times of a random walk on the prism graph, Möbius ladder graph, and the generalized Petersen graph $\mathrm{GP}(n,k)$ (see Figures 15, 16, and 19 respectively) as well as their triangulated versions (see Figure 14), and finish by giving some possible directions for future research.

1.1. **Markov Chain Theory.** We first introduce some basic concepts on Markov chains. We will be working with a discrete probability space throughout.

Definition 1.1. We call a sequence of random variables $\{X_i\}_{i=0}^{\infty}$ on a common state space Ω a **Markov chain** if it satisfies the **Markov property**; that is, for a probability measure $\mathbf{P}$ on Ω, we have

$$\mathbf{P}\{X_n = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}\} = \mathbf{P}\{X_n = y \mid X_{n-1} = x_{n-1}\}.$$

Informally, this means that the probability of a future event happening depends only on the information of the current event. Put into the context of a random walker, if the $\{X_i\}_{i=0}^{\infty}$ are random variables denoting the location of the random walker, then the Markov property states that the probability of where the walker will go in the next step depends only on where the walker is now, and not how the walker got there.

Remark 1.2. Sometimes the indices will be dropped when referring to Markov chains, e.g. we will write $\{X_i\}$ instead of $\{X_i\}_{i=0}^{\infty}$.

Definition 1.3. We define a transition matrix to be a matrix P such that

$$P(x,y) = \mathbf{P}\{X_n = y \mid X_{n-1} = x_{n-1}\},$$

where $X_i \in \{X_i\}_{i=0}^{\infty}$.

Proposition 1.4. We have that $P^t(x,y)$, $t > 0$, is the probability of going from state x to state y in t steps.

Proof. We give a sketch of the proof here, proceeding by induction. Notice that it holds for $t = 1$ by definition. We will show the case $t = 2$ for clarity. In this case, we have the probability of going from a state x to a state y in two steps is the same as going from a state x to any intermediate state $z \in \Omega$ in one step, and then from state z to state y. Since it could be any intermediate state, we take a union over these probabilities; that is,

$$\{X_2 = y \mid X_0 = x\} = \bigcup_{z \in \Omega} \{X_1 = z \mid X_0 = x\} \cap \{X_2 = y \mid X_1 = z\}.$$

Notice that $\{X_1 = z \mid X_0 = x\}$ is independent of $\{X_2 = y \mid X_1 = z\}$ under our probability measure. We apply our probability measure to both sides to get

$$\mathbf{P}\{X_2 = y \mid X_0 = x\} = \sum_{z \in \Omega} \mathbf{P}\{X_1 = z \mid X_0 = x\}\mathbf{P}\{X_2 = y \mid X_1 = z\}.$$

However, we can use our transition matrix notation; rewriting the right hand side, we have

$$\mathbf{P}\{X_2 = y \mid X_0 = x\} = \sum_{z \in \Omega} P(x,z)P(z,y) = P^2(x,y)$$

by matrix multiplication, as desired. Now, assume the statement holds for $d > 0$. That is,

$$\mathbf{P}\{X_d = y \mid X_0 = x\} = P^d(x,y).$$

We want to then show the statement holds for $d+1$. However, this is analogous to the arugment for $t=2$, and so we end up with

$$\begin{aligned}\mathbf{P}\{X_{d+1}=y \mid X_0=x\} &= \sum_{z\in\Omega}\mathbf{P}\{X_{d+1}=y, X_d=z \mid X_0=x\}\\ &= \sum_{z\in\Omega}P^d(x,z)P(z,y)=P^{d+1}(x,y).\end{aligned}$$

■

Remark 1.5. Since we have that $P^t(x,y)$ is the probability of going from x to y in t steps, this shows that the transition matrix encodes all of the information of the Markov chain.

Remark 1.6. Throughout, we take Ω to be the state space for our Markov chain and P to be its transition matrix unless stated otherwise.

Definition 1.7. We say that a Markov chain is **irreducible** if for all $x,y\in\Omega$ we have some $0\le r<\infty$ such that $P^r(x,y)<\infty$.

In other words, it is irreducible if it is possible for the Markov chain to reach every state from every state in a finite number of steps.

Definition 1.8. We define the **period** of a state $x\in\Omega$ to be

$$\mathscr{T}(x) := \gcd\{t\ge 1 \mid P^t(x,x)>0\}.$$

In other words, the period is the greatest common divisor of the set of times where x has a non-trivial probability of returning to x.

Definition 1.9. We say that a chain is **aperiodic** if $\mathscr{T}(x)=1$ for all $x\in\Omega$, and we say the chain is **periodic** otherwise.

Definition 1.10. We say that a distribution, which will be a row vector, π on Ω is a **stationary distribution** if it satisfies

$$\pi P=\pi.$$

Definition 1.11. We say $\hat{\pi}$ is a **limiting distribution** if it is a distribution on Ω and

$$\lim_{t\to\infty}P^t(x,y)=\hat{\pi}(y).$$

The reason we care about periodicity and irreducibility is due to the following theorem.

Theorem 1.12. If a Markov chain is irreducible and aperiodic, then it has a unique stationary distribution which is also its limiting distribution.

Proof. This follows from Corollary 1.17 and Theorem 4.9 in [3]. ■

While the proof of the above theorem is involved, we give some intuition as to why you need these conditions. If your Markov chain was not irreducible, then that means there are two separate components. As a result, if there even is a stationary distribution, it will not be unique in any sense as there will be a stationary distribution for each irreducible component. Aperiodicity is important as it ensures that our stationary distribution is the limiting distribution. Imagine the Markov chain on the cycle $\mathbb{Z}/4\mathbb{Z}$ which moves left with probability $1/2$ and right with probability $1/2$. We see that there is no limiting distribution, as its distribution relies on whether or not its on an even or odd number. Thus, we really do need both conditions for this to hold.

Throughout, we will assume all of our Markov chains will be aperiodic and irreducible so that they will have unique stationary distributions that also are their limiting distributions. The stationary distribution will really be how we measure "sufficiently random." Taking our Markov chain to be aperiodic and irreducible, we have that in the long run it will be close to this stationary distribution. Referring back to the shuffling example, our stationary distribution is the uniform distribution on all possible configurations of the deck, and so being close to the stationary distribution means being close to having a fair deck.

We now need to formalize the notion of distance between two distributions. We define the **total variation distance** between probability distributions μ and ν on Ω to be

$$||\mu - \nu||_{TV} = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|.$$

Remark 1.13. We can equivalently define total variation distance to be

$$||\mu - \nu||_{TV} = \frac{1}{2} \sum_{x \in \Omega} \big|\mu(x) - \nu(x)\big| = \sum_{\substack{x \in \Omega \\ \mu(x) \geq \nu(x)}} \big(\mu(x) - \nu(x)\big).$$

The proof of the equivalence follows from Proposition 4.2 in [3]. A good visualization of total variation distance can be seen in Figure 4.1 in [3].

Remark 1.14. It can be shown that total variation distance is a metric, and so really does satisfy the intuition of distance. With the above equivalence, we also see that it is closely related to the L^1 norm.

In particular, we would like to study the position of a random walker in comparison to its stationary distribution. As a result, we define

$$d(t) := \max_{x \in \Omega} ||P^t(x, \cdot) - \pi||_{TV}$$

to be the distance between $P^t(x, \cdot)$, the random walker which is starting at x whose position is $\cdot$ at time t, and π. Referring back to the example of shuffling decks, this tells us how close our deck, starting at configuration x, is to stationarity after t shuffles. We would like to know roughly when P^t is close to π; that is, when their total variation distance is within $\epsilon > 0$. This notion makes sense as $d(t)$ is decreasing as t increases.

Definition 1.15. We define the **mixing time** to be

$$t_{\text{mix}}(\epsilon) := \min\{t \mid d(t) \leq \epsilon\}.$$

Understanding the mixing time of random walks is central to our project, and as a result we would like to be able to get bounds on the mixing time. One method for bounding the mixing time is coupling, which we use throughout.

Definition 1.16. We define a **coupling** of two probability distributions μ and ν to be a pair of random variables (X, Y) defined on a single probability space Ω such that

$$\sum_{y \in \Omega} \mathbf{P}\{X = x, Y = y\} = \mathbf{P}\{X = x\} = \mu(x)$$

and

$$\sum_{x \in \Omega} \mathbf{P}\{X = x, Y = y\} = \mathbf{P}\{Y = y\} = \nu(y).$$

That is, a coupling of probability distributions instills a set of rules which determine how the two probability distributions behave relative to one another. However, if we viewed just one probability distribution without the other, we would see that it still satisfies being a probability distribution.

Definition 1.17. We define a **coupling of Markov chains** with transition matrix P to be a process $(X_t, Y_t)_{t=0}^{\infty}$ with the property that both $\{X_t\}_{t=0}^{\infty}$ and $\{Y_t\}_{t=0}^{\infty}$ are Markov chains with transition matrix P.

For example, think of the random walker starting at location x, and the same random walker except starting at location y. Then they share a transition matrix, P, are both Markov chains, and we can define a set of rules that each walker must adhere to relative to the other.

Definition 1.18. Given a Markov chain on Ω with transition matrix P, we define a **Markovian coupling** of two P-chains to be a **Markov chain** $\{(X_t, Y_t)\}_{t=0}^{\infty}$ with state sapce $\Omega \times \Omega$ which satisfies, for all x, y, x', y'

$$\mathbf{P}\{X_{t+1} = x' \mid X_t = x, Y_t, y\} = P(x, x')$$
$$\mathbf{P}\{Y_{t+1} = y' \mid X_t = x, Y_t = y\} = P(y, y').$$

In general, we also require that if $X_t = Y_t$ for some $t \geq 0$, then we have $X_s = Y_s$ for all $s \geq t$. That is, once they have coalesced, they stay coalesced.

Markovian couplings are nice in the sense that they are easy to describe. However, Markovian couplings will not always give you the best upper bound. All couplings in this paper will be Markovian.

We now need to connect having information on two random walkers, $P^t(x, \cdot)$ and $P^t(y, \cdot)$, to information on the random walker compared to the stationary distribution.

Proposition 1.19. If we define

$$\bar{d}(t) := \max_{x,y\in\Omega} ||P^t(x, \cdot) - P^t(y, \cdot)||_{TV},$$

then we have

$$d(t) \leq \bar{d}(t).$$

Proof. A proof is given in [3]. We will give a sketch here. Since π is a stationary distribution, we have

$$\pi P = \pi$$

gives us

$$\sum_{x\in\Omega} P(x, y)\pi(x) = \pi(y)$$

by matrix multiplication. Furthermore, we have

$$\sum_{x\in\Omega} P(x, A)\pi(x) = \pi(A)$$

for any subset $A \subseteq \Omega$. Now, notice that

$$|P^t(x, A) - \pi(A)| = \left| P^t(x, A) - \sum_{y\in\Omega} P^t(y, A)\pi(y) \right|.$$

Since π is a probability distribution, we have

$$\sum_{y\in\Omega} \pi(y) = 1,$$

and so we get

$$\left| P^t(x, A) - \sum_{y\in\Omega} P^t(y, A)\pi(y) \right| = \left| \sum_{y\in\Omega} \pi(y)\big(P^t(x, A) - P^t(y, A)\big) \right|.$$

By the triangle inequality, this is less than or equal to

$$\sum_{y \in \Omega} \pi(y) \big| P^t(x, A) - P^t(y, A) \big|.$$

This is a weighted average, and so in particular we have

$$\sum_{y \in \Omega} \pi(y) \big| P^t(x, A) - P^t(y, A) \big| \leq \max_{y \in \Omega} \big| P^t(x, A) - P^t(y, A) \big|.$$

This gives us

$$d(t) \leq \bar{d}(t).$$

■

We see that studying $\bar{d}(t)$ gives us information on $d(t)$, which we can then use to derive the mixing time. We can then use the following theorem to get information on bounding $\bar{d}(t)$, and thus on bounding $d(t)$.

Theorem 1.20. For any two Markov chains $\{X_i\}$ and $\{Y_i\}$ over a common state space Ω with $X_0 = x$, $Y_0 = y$, let τ_{couple} be the time of the chains coalesce. That is,

$$\tau_{\text{couple}} := \min\{t \mid X_s = Y_s \text{ for all } s \geq t\}.$$

Then

$$d(t) \leq \bar{d}(t) \leq \max_{x,y \in \Omega} \mathbf{P}\{\tau_{\text{couple}} > t \mid X_0 = x, Y_0 = y\} \leq \max_{x,y \in \Omega} \frac{\mathbf{E}(\tau \mid X_0 = x, Y_0 = y)}{t}.$$

Proof. This a consequence of Corollary 5.5 in [3] and Proposition 1.19, with the last inequality coming from Markov's inequality. ■

We will often want to transform our complicated Markov chain to a much simpler one. We formalize this in the definition below.

Definition 1.21. Let $\{X_t\}$ be a Markov chain. Then the **quotient Markov chain** $\{X'_t\}$ is the corresponding Markov chain quotiented out by some equivalence relation between states. That is, there's a bijective function between $\{X_t\}$ and $\{X'_t\}$ which preserves transition probabilities.

An example of this can be seen in the proof of Proposition 3.6. More examples of Markov chains and couplings can be found in the Appendix (Section 5).

1.2. **Graph Theory.** We now introduce some concepts from graph theory.

Definition 1.22. We define a **graph** G = (V,E) to consist of a set of vertices V and a set of edges $E \subseteq V \times V$.

In other words, the vertices represent some collection of objects and the edges represent some relation between those objects. We can visually represent graphs by having the vertices be dots and the edges be lines between the dots.

Remark 1.23. We will be working with **undirected graphs**, or graphs where if $(x, y) \in E$ then $(y, x) \in E$ as well. A **directed graph** is a graph such that $(x, y) \in E$ does not imply $(y, x) \in E$. We will also, for the most part, be working with **simple graphs**, or graphs that do not have any kind of self-loops. Notationally, we have $(x, x) \notin E$ for all $x \in V$.

In general, undirected graphs are drawn with just lines as edges (see Figure 1), and directed graphs are drawn with arrows for edges (see Figure 2).

Remark 1.24. Notice that a Markov chain is a **weighted digraph**, or a directed graph whose edges have some sort of value, and so we graphically represent Markov chains using directed graphs.

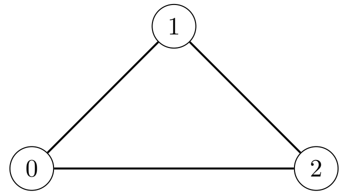

FIGURE 1. The undirected graph with $V = \{0, 1, 2\}$, $E = \{(0, 1), (1, 0), (1, 2), (2, 1), (0, 2), (2, 0)\}$.

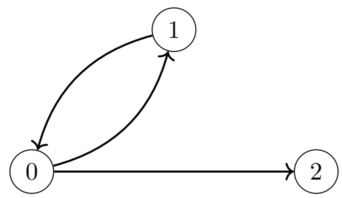

FIGURE 2. The directed graph with $V = \{0, 1, 2\}$, $E = \{(0, 1), (1, 0), (0, 2)\}$.

Definition 1.25. If $(x, y) \in E$, we say that x and y are **neighbors**.

Definition 1.26. Let

$$N(x) := \{y \in \Omega \ : \ (x, y) \in E\}$$

be the set of neighbors for some vertex x. Then we say that the **degree** of x is

$$\deg(x) := |N(x)|,$$

or the number of neighbors it has.

Definition 1.27. Say we have a graph G. We define a **simple random walk** on G to be the Markov chain with state space $\Omega = V$ and transition matrix

$$P(x, y) = \begin{cases} \frac{1}{\deg(x)} & \text{if } (x, y) \in E \\ 0 & \text{otherwise.} \end{cases}$$

In other words, a random walker will uniformly select a neighbor of the vertex it is at and move accordingly. We note that the simple random walk on G is not always irreducible or aperiodic. If our graph is **connected**, that is, there exists some path of edges such that x can get to y for all $x, y \in V$, then the simple random walk will be irreducible. To solve the issue of periodicity, we will make our simple random walk **lazy**. To do so, we simply add a $1/2$ chance to stay in place.

Definition 1.28. We have a **lazy random walk** if our transition matrix is then

$$P(x, y) = \begin{cases} \frac{1}{2} & \text{if } x = y \\ \frac{1}{2\deg(x)} & \text{if } (x, y) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.29. We say that a graph is **s-regular** if $\deg(x) = s$ for all $x \in V$.

Definition 1.30. We define the **complete graph on n-vertices** to be the graph where every vertex is connected to every other vertex, and denote it by K_n.

Definition 1.31. If we have graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then **gluing** G_1 to G_2 along vertices $x \in V_1$ and $x' \in V_2$ involves identifying x and x' together,

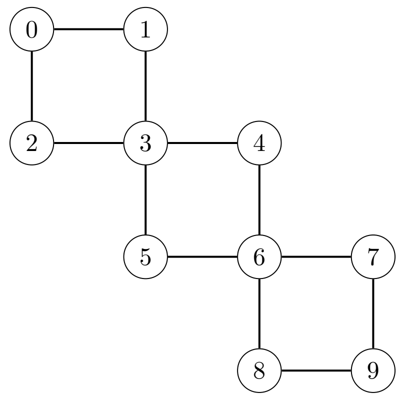

FIGURE 3. Diagonal gluing with $n = 4$, $k = 3$.

and then attaching all of the edges which are connected to x and x' to this new vertex.

Definition 1.32. We define **triangulating** a graph to be replacing each vertex with a copy of $C(3)$ and appropriately attaching edges to corresponding vertices. More can be seen in Section 4, as well as in Figure 14.

Definition 1.33. We define the **graph metric** ρ to be the function which takes two vertices $v_1, v_2 \in V$ and measures the **distance** between them; in other words, the minimum number of edges one must take to get from one vertex to another.

2. Diagonal Gluings of 2-Regular Graphs

We first define a gluing process on 2-regular graphs. Start with a 2-regular graph of size n, and identify two vertices v_1, v_2 such that $\rho(v_1, v_2) = 2$ to be the corners. At step $k = 2$, glue a copy of the 2-regular graph along one of the corners, and then identify a vertex v_3 to be the vertex on the new graph such that $\rho(v_2, v_3) = 2$. Iterate this process for each remaining k; that is, identify a vertex to be the corner and glue along that. An example is given in Figure 3. We will use the standard lazy random walk on this graph. In other words, we define the Markov chain on this graph as follows:

$$P(x, y) = \begin{cases} \frac{1}{4} & \text{if } y \in N(x) \text{ and } \deg(x) = 2, \\ \frac{1}{8} & \text{if } y \in N(x) \text{ and } \deg(x) = 4, \\ \frac{1}{2} & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Considering even n leads us to the following proposition.

Proposition 2.1. Fix n even and any k for the proposed structure. Let $\{X_i\}$ and $\{Y_i\}$ be two Markov chains with transition matrix P such that they are both lazy random walks on the graph. We have a coupling such that if τ is the time until they coalesce, then

$$\max_{x,y \in \Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{(kn)^2}{2}.$$

In particular, we see

$$t_{\text{mix}}(\epsilon) \leq \frac{(kn)^2}{2\epsilon}$$

Proof. We do a sort of depth-first search approach to our coupling. Let 0 be the top left most node (i.e. the vertex on the original graph which as at maximal distance from the glued vertex) and define $h : \Omega \to \{0, \dots, (kn)/2\}$ to be the function

$$h(v) = \rho(v, 0).$$

In other words, h measures the relative 'height' of a random walker. Let X_i and Y_i be our random walkers on this graph. We see that the maximum distance will be $(kn)/2$, since the maximal distance on the 2-regular graph of size n is $n/2$ and we are simply increasing this distance k times. We can divide up our vertices into classes based on their height, and since the vertices in these corresponding height classes will have the same transition probabilities between other height classes, we can push this to a quotient Markov chain. Thus, we can now think of this as a random walk on the chain $\{0, \dots, (kn)/2\}$ with the transition probabilities

$$P(x, y) = \begin{cases} \frac{1}{4} & \text{if } y \in N(x) \text{ and } x \neq 0 \text{ nor } \frac{kn}{2}, \\ \frac{1}{2} & \text{if } y \in N(x) \text{ and } x = 0 \text{ or } \frac{kn}{2}, \\ \frac{1}{2} & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Our coupling is now the following: since we can just use the Markov chain given above to model where our walkers are, we couple so that they both move the same direction. If they were to try to move in a direction that doesn't exist at the end points, we just have the walker wait in place. This will preserves the property that $h(X_s) \le h(Y_s)$, assuming without loss of generality that X_s is closer to 0. To see how long it takes for the walkers to coalesce, we just need to measure the maximum expected amount of time it takes for a walker to reach 0. This follows since this is the same as measuring how long it takes Y_s to reach 0, and since $h(X_s) \le h(Y_s)$ we get $h(X_s) = h(Y_s) = 0$. Let τ' be the amount of time it takes a walker $\{X_t\}$ to reach 0. We set up a series of functions $f_j = \mathbf{E}(\tau' \mid X_0 = j)$ such that $f_0 = 0$,

$$f_j = \frac{1}{4}(1 + f_{j-1}) + \frac{1}{2}(1 + f_j) + \frac{1}{4}(1 + f_{j+1})$$

for $0 < j < (kn)/2$, and

$$f_{(kn)/2} = \frac{1}{2}(1 + f_{(kn)/2}) + \frac{1}{2}(1 + f_{(kn)/2-1}).$$

Claim 2.2. For $0 < j < (kn)/2$,

$$f_j = 2j + \frac{j}{j+1} f_{j+1}.$$

Proof. We proceed by induction. By substitution, we have

$$f_1 = 2 + \frac{1}{2} f_2,$$

and so the base case holds. Now assume Claim 2.2 holds for some $1 < k < n/2 - 2$. We want to then show the induction hypothesis holds for $k + 1$. By the above relations, we get

$$f_{k+1} = \frac{1}{4}(1 + f_k) + \frac{1}{2}(1 + f_{k+1}) + \frac{1}{4}(1 + f_{k+2}).$$

Rearranging this, we have

$$f_{k+1} = \frac{(3\,k + 2)\, f_{k+1} + 2\,(k + 1)\,(k + (1/2)\, f_{k+2} + 2)}{4\,k + 4}.$$

Solving this for f_{k+1} gives

$$f_{k+1} = 2(k + 1) + \frac{k + 1}{k + 2} f_{k+2}$$

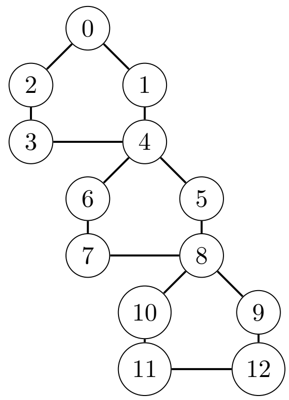

FIGURE 4. Diagonal gluing with $n = 5$, $k = 3$.

as desired. ■

Claim 2.3. With this construction, we have $f_i < f_{i+1}$ for all $0 \leq i \leq (kn)/2$, and so $f_{(kn)/2}$ is the maximum over the set of f_i.

Proof. It is a result of Claim 2.2 and the construction. ■

With these claims, we get

$$f_{(kn)/2-1} = kn - 2 + \frac{kn-2}{kn} f_{(kn)/2}.$$

Substituting this into

$$f_{(kn)/2} = \frac{1}{2}\left(1 + f_{(kn)/2}\right) + \frac{1}{2}(1 + f_{(kn)/2-1}$$

and solving gives

$$f_{(kn)/2} = \frac{(kn)^2}{2},$$

which gives us that

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{(kn)^2}{2}.$$

■

For odd n, the process is not as easy. The even case really allowed us to exploit the fact that both paths leading out of the glued or corner vertex to the next glued or corner vertex were equidistant. With odd n, we have that one of the paths is longer than the other one. In order to remediate this, we take a much more constructed approach. We will focus on the case of $n = 5$, although this procedure can be modified for all odd n. For a visual example of the gluing procedure, see Figure 4.

Instead of using the same transition probabilities as before, we use a **modified lazy random walk** on the Markov chain.

Definition 2.4. The **modified lazy random walk** is the lazy random walk with transition probabilities modified so that the stationary distribution is uniform.

For the diagonally glued graph, the modified lazy random walk will have transition probabilities

$$P(x,y) = \begin{cases} \frac{1}{8} & \text{if } y \in N(x), \\ \frac{1}{8}\big(4 - |N(x)|\big) + \frac{1}{2} & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5. We now consider $n = 5$. We have a coupling so that if τ is the amount of time it takes two walkers to coalesce, then

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq 24k^2 - \frac{528}{25}k + \frac{25}{4}.$$

Furthermore,

$$t_{\mathrm{mix}}(\epsilon) \leq \frac{24k^2}{\epsilon} - \frac{528}{25\epsilon}k + \frac{25}{4\epsilon}.$$

Proof. We first define the coupling procedure. Take two random walkers, denoted by $\{X_t\}$ and $\{Y_t\}$, and let them move randomly until they are coupled at the same relative position on the pentagon. To be precise on same relative position, we will need to outline a labeling procedure. For $k \geq 2$, label each corner vertex by multiples of 4 increasing. Arbitrarily choose some vertex distance 2 away from the first corner vertex (the corner vertex we labeled as 4) to be 0. Label the vertex adjacent to both 0 and 4 as 1. Along the other path to 4, label the vertex closest to 0 as 1 and the vertex closest to 4 which has not been labeled as 3. Follow the same procedure for the remaining pentagons; that is, for vertices between $4t$ and $4(t+1)$, label the vertex distance one from both $4t$ and $4(t+1)$ as $4t+1$, label the vertex closest to $4t$ thats not labeled as $4t+2$, and the remaining vertex on the pentagon as $4t+3$. Then two random walkers are on the same relative position if the labels of the vertices they are on are in the same class modulo 4.

We will define h as in Proposition 2.1. That is,

$$h(v) = \rho(v, 0).$$

We assume again that $h(X_t) \leq h(Y_t)$ without loss of generality. Notice, however, we cannot do the same quotient Markov chain argument as before. This is because vertices which are within the same height class will not necessarily have the same transition probabilities to other vertices in other height classes. As a result, we have to preform a different sort of procedure. Instead, we will have that the walkers will then move in the same directions until we have $h(X_t) = 0$ or until $h(Y_t) = 4k$.

From there, we shift our coupling to a different kind of coupling. This coupling will have a few different cases: if the walkers are in the same position, they coalesce and from then on they move the same direction; if the walkers are in the same position relative to the pentagon (i.e., they are in the same class determined by their distances away from both glued points), then they move together relative to their respective pentagons; if they are in different classes, we have that they move according to the table in Figure 5 (for simplicity, if there are not enough rows for the corresponding paths then this means just repeat staying at the respective vertex). For vertices beyond 7, using the labeling procedure outlined in the first paragraph, take your vertices labels modulo 4 and proceed according to the table (if your vertex is congruent to 0 modulo 4 and is a glued vertex, then treat it like 4).

The reason for defining the coupling procedure this way is to preserve the property that $h(X_s) \leq h(Y_s)$. This allows us to focus on one walker, which is simpler than trying to focus on both. This leads us to the following proposition.

(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
(1,1)	(0,0)	(1,4)	(0,1)	(0,0)	(0,2)	(0,1)	(3,3)	(0,1)	(2,1)
(0,0)	(2,2)	(2,2)	(0,3)	(4,3)	(1,3)	(4,4)	(2,2)	(3,3)	(4,4)
(2,4)	(1,3)	(0,3)	(0,4)	(1,2)	(4,4)	(1,3)	(0,4)	(2,4)	(3,3)
(0,1)	(0,2)	-	(1,5)	-	-	(1,5)	(2,3)	(2,5)	(3,5)
-	-	-	(2,6)	-	-	(1,6)	-	(2,6)	(3,6)

(4,5)	(4,6)	(4,7)
(1,4)	(5,7)	(5,8)
(6,8)	(6,6)	(6,6)
(5,5)	(1,6)	(3,7)
(4,5)	(3,4)	(1,7)
-	-	(4,7)

FIGURE 5. Locations and destinations of (X_t, Y_t) on the pentagon.

Claim 2.6. For the coupling described above, once $X_s = 0$ and $Y_t = 4j$ for $0 < j < k-1$, we get $h(X_s) \leq h(Y_s)$ for all $s \geq t$. In particular, we get $X_s = Y_s$ once $h(Y_s) = 0$.

Proof. We show a similar statement; that is, if $h(X_k) \leq h(Y_k)$ for all $k \leq t$, then $h(X_{t+1}) \leq h(Y_{t+1})$ for all possible choices of X_{t+1}, Y_{t+1}. We proceed by induction. The base case of $X_t = 0$ and $Y_t = 4j$ for $0 < j < K$ is true. Assume that s is the first instance where this does not hold; that is, $h(X_t) \leq h(Y_t)$ for all $t \leq s$, but $h(X_{s+1})$ may be larger than $h(Y_{s+1})$. While one would normally need to consider many different cases, using Figure 5 and the coupling procedure outline above, we see that the only position where $h(X_t) \leq h(Y_t)$ but $h(X_{s+1})$ may be larger on the first pentagon is at $(4,3)$ (read $X_s = 4$, $Y_s = 3$). The strategy will be to work backwards and show that it is actually impossible to reach this position from the starting configuration.

Working backwards, we see that $X_{s-1} = 1$ or 3 and $Y_{s-1} = 2$ or 4. If $X_{s-1} = 1$, then we have $(1,2)$ or $(1,4)$ as options. If the walkers are at $(1,2)$, we see that they either coalesce or they go to $(4,3)$, and so this is a possibility. We see that at $(1,4)$ they cannot reach $(4,3)$, and so this is fine. We see that $(3,2)$ results in a contradiction, and so we ignore this case. This leaves $(3,4)$ as an option, but as we can see from the table this does not result in $(4,3)$ and so we exclude it.

In order for $X_{s-1} = 1$, we need $X_{s-2} = 0$ or $X_{s-2} = 4$, and likewise for $Y_{s-1} = 2$ we need $Y_{s-2} = 0$ or $Y_{s-2} = 3$. At $(0,0)$, the walkers are coalesced and so it's impossible to reach $(1,3)$. At $(0,3)$, we see that according to the table it is impossible for us to move to $(1,2)$, and so we omit it. The case $(4,2)$ again results in a contradiction. $(4,3)$ also results in a contradiction, since this implies that they were at this place before s. Thus, we see that at the start pentagon it is impossible to reach the case $(4,3)$.

We now consider a pentagon between two glued points. We see that we run into the same issue. Using Figure 4, we have that $X_{s-1} = 8$ and $Y_{s-1} = 7$ leads to a possibility of $h(X_s) > h(Y_s)$. However, for this to happen, we would need $X_{s-2} = 7$ or $X_{s-2} = 5$. If $X_{s-2} = 7$, this means that $Y_{s-2} = 6$ or $Y_{s-2} = 8$ in order to have it at 7 at time $s-1$. The case of $Y_{s-2} = 6$ immediately gives us a contradiction, since this implies $h(Y_{s-2}) < h(X_{s-2})$. If $Y_{s-2} = 8$ and $X_{s-2} = 7$, then we see that the walkers are at $(7, 8)$, which we treat as $(3, 4)$. In such a case, it is impossible to get $(8, 7)$ on the next step.

In the other direction, if $X_{s-2} = 5$, then we have either $Y_{s-2} = 6$ or $Y_{s-2} = 8$. Thus, we are at either $(5, 6)$ or $(5, 8)$. If we're at $(5, 6)$, we treat it as $(1, 2)$, and so we see that it is possible to reach $(8, 7)$ according to the table. If the walkers are at $(5, 8)$, we see that it is impossible; treating this as $(1, 4)$, we have that they must coalesce or we're at $(5, 7)$ at $s - 1$, which prevents this case.

We then see that in order to get $X_{s-2} = 5$ and $Y_{s-2} = 6$, we must have either $X_{s-3} = 4$ or $X_{s-3} = 8$ and $Y_{s-3} = 7$ or $Y_{s-3} = 4$. We notice that by the table, it is impossible to reach $(5, 6)$ from $(4, 7)$, and so we are fine. If $X_{s-3} = Y_{s-3} = 4$, then it is also impossible, and so we're fine in such a case. If $X_{s-3} = 8$ and $Y_{s-3} = 4$, we run into a contradiction and so we stop. Thus, we have to consider the case $(8, 7)$. But this is itself a contradiction, since we assume that it was impossible to have reached this state prior.

Since we have that it holds for the first pentagon, and for all the glued pentagons, the only other case to consider is at the opposite end. However, for the opposite end, we notice that by flipping the coupling we get that Y_s will always be closer to the opposite end, giving us the same result. Thus, it holds for both end pentagons and all other glued pentagons, and so we see that we get $h(X_s) \leq h(Y_s)$ for all $t \geq s$. As a result, once we get $h(Y_s) = 0 = h(X_s)$, we get that they have coalesced. ■

With τ being the amount of time it takes until they coalesce, we set $\tau = \tau_1 + \tau_2$, where τ_1 is the amount of time it takes for them to couple in their respective pentagons and τ_2 is the amount of time it takes for $h(Y_s) = 0$. Using Gambler's ruin (Proposition 5.1) to bound τ_1 above, we have that this is bounded by a constant,

$$\max_{x,y \in \Omega} \mathbf{E}(\tau_1 \mid X_0 = x, Y_0 = y) \leq \frac{25}{4}.$$

The bound for τ_2 is a little more difficult. Using the same labeling strategy as outlined in Figure 4 and the prior proof, we find equations for $\mathbf{E}(\tau_2 \mid Y_0 = j)$ where $j \equiv 1 \pmod 4$ and $j \equiv 3 \pmod 4$. Let $f_i = \mathbf{E}(\tau_2 \mid Y_0 = i\}$.

Claim 2.7. Let g_i for $0 < i < k + 1$ be the equations for $\mathbf{E}(\tau_2 \mid Y_0 = j\}$, $j \equiv 1 \pmod 4$, where i here represents which pentagon we're looking at, with the origin corresponding to $i = 1$ and the final corresponding to $i = k$. Then we have

$$g_i = \frac{68 + 48(i - 2)}{5} + \frac{2i - 1}{2i} f_{4i}$$

for $1 < i < k$ and

$$g_1 = 4 + \frac{1}{2} f_4.$$

Likewise, let h_i for $0 < i < k + 1$ be the formula for $\mathbf{E}(\tau_2 \mid Y_0 = j\}$, $j \equiv 3 \pmod 4$, where i here represents which pentagon we're looking at, with the origin corresponding to $i = 1$ and the final corresponding to $i = k$. Then we have

$$h_i = \frac{72 + (32)(i - 2)}{5} + \frac{3i - 1}{3i} f_{4i}$$

for $1 < i < k$ and

$$h_1 = 8 + \frac{2}{3} f_4.$$

Proof. We see that this holds for $i = 1, 2$ respectively by just plugging these equations in. We preform again an inductive argument. Assuming Claim 2.7 holds for

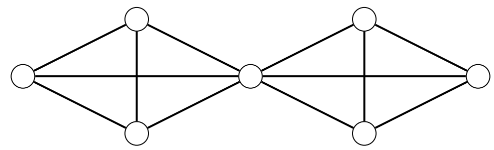

Figure 6. Gluing of two complete graphs along a single vertex, with $n = 4$.

i again, we show that the inductive hypothesis holds for $i+1$. Notice that by the recursive assignment we get

$$\begin{aligned} f_{4i} &= \frac{1}{8}(1+g_i) + \frac{1}{8}(1+h_i) + \frac{1}{8}(1+f_{4i+2}) + \frac{1}{8}(1+g_{i+1}) + \frac{1}{2}(1+f_{4i}), \\ g_{i+1} &= \frac{1}{8}(1+f_{4i}) + \frac{1}{8}(1+f_{4(i+1)}) + \frac{3}{4}(1+g_{i+1}), \\ f_{4i+2} &= \frac{1}{8}(1+f_{4i}) + \frac{1}{8}(1+h_{i+1}) + \frac{3}{4}(1+f_{4i+2}), \\ h_{i+1} &= \frac{1}{8}(1+f_{4i+2}) + \frac{1}{8}(1+f_{4(i+1)}) + \frac{3}{4}(1+h_{i+1}). \end{aligned}$$

Solving these equations then gives us the desired result, showing that Claim 2.7 indeed holds. ∎

Using this, we can solve for when our walker is in the opposite corner, which will give us the maximum expected value. Doing so gives us

$$\max_{x,y\in\Omega} \mathbf{E}(\tau_2 \mid X_0 = x, Y_0 = y) \le 24k^2 - \frac{528}{25}k.$$

Summing results together gives us

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) = \max_{x,y\in\Omega} \mathbf{E}(\tau_1+\tau_2 \mid X_0 = x, Y_0 = y) \le 24k^2 - \frac{528}{25}k + \frac{25}{4}.$$

∎

Remark 2.8. The method we used to find the mixing time of the diagonally glued pentagons is not necessarily unique to the pentagon. However, it is not clear how to sufficiently generalize these results so we can find a mixing time for all $n > 5$ without manually going through and preforming the calculation. Further work could explore this, as well as just exploring the case where we use the standard lazy random walk. One might also want to see if there is a completely general coupling on both n even and n odd, giving us a good way to asymptotically measure the mixing time.

3. Gluings of Complete Graphs

Suppose we took two complete graphs of size n and glued them along a single vertex. An example of this can be seen in Figure 6. The mixing time of this graph is explored using strong stationary times in [3] in Example 6.5.1. We do a similar analysis here using coupling.

Proposition 3.1. The mixing time of the lazy random walk on the graph obtained from gluing two complete graphs along a single vertex is bounded above by

$$t_{\text{mix}}(\epsilon) \le \frac{4n}{\epsilon}.$$

Furthermore, we see that this is actually the same if we glue $k \ge 2$ complete graphs along the same vertex.

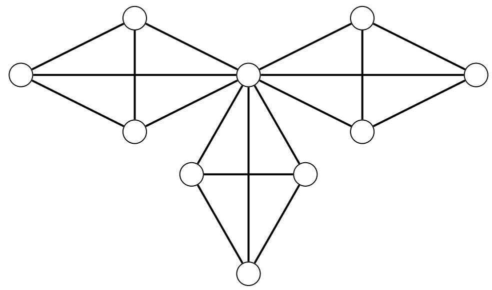

FIGURE 7. Gluing complete graphs along a single vertex, with $n = 4$, $k = 3$.

Proof. First, couple our walkers in their respective complete graphs as we did with the pentagon example. That is, we want them to be in the same position on their respective graphs. Flip a fair coin and move the walkers $\{X_n\}$ and $\{Y_n\}$ according to the coin flip. Then the probability distribution of the walkers being on the same spot is geometric, with probability $1/(2n)$ of success. Let τ_1 be the amount of time it takes for the walkers to coalesce in this setting. Then since it's geometric, we have

$$\max_{x,y\in\Omega} \mathbf{E}(\tau_1 \mid X_0 = x, Y_0 = y) \leq 2n.$$

Now, we shift our coupling so that the walkers always move to the same location on their respective graphs. We have that they coalesce when they both hit the center node; that is, the node which all the complete graphs are glued along. The distribution is again geometric, and again we have probability $1/(2n)$ of success. Let τ_2 be the amount of time it takes for the walkers to coalesce here. We get now

$$\max_{x,y\in\Omega} \mathbf{E}(\tau_2 \mid X_0 = x, Y_0 = y) \leq 2n.$$

Now, let $\tau = \tau_1 + \tau_2$ be the amount of time it takes for the walkers to coalesce entirely. Using the above results, we get

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) = \max_{x,y\in\Omega} \mathbf{E}(\tau_1 + \tau_2 \mid X_0 = x, Y_0 = y) \leq 4n,$$

giving us

$$t_{\text{mix}}(\epsilon) \leq \frac{4n}{\epsilon}.$$

Notice that this same argument works for when we have $k \geq 2$ complete graphs glued along the same vertex (see Figure 7). We have then that the total mixing time is

$$t_{\text{mix}}(\epsilon) \leq \frac{4n}{\epsilon}.$$

■

We diverge slightly to discuss a coupling on a specific version of a tree. Consider the tree with depth k and three possible paths leading out of its root node v_*, and each of the non-vertex non-boundary vertices have degree 2. An example of this can be seen in Figure 8. We would like to understand the mixing time on this.

Proposition 3.2. For the above scenario, we have

$$t_{\text{mix}} \leq \frac{3k(k+1)}{\epsilon}.$$

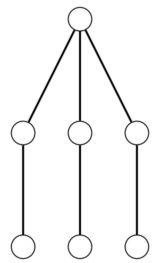

FIGURE 8. Tree with three paths of length $k = 2$.

Proof. First, we couple based on height. Here, height will be defined in the standard way; that is, $h(v) = \rho(v, v_*)$. We'll set our transition probabilities to be

$$P(x,y) = \begin{cases} \frac{1}{2} & \text{if } x = y = v_*, \\ \frac{2}{3} & \text{if } x = y \neq v_* \text{ and } h(x) \neq k, \\ \frac{5}{6} & \text{if } x = y,\ h(x) = k, \\ \frac{1}{6} & \text{if } y \in N(x), \\ 0 & \text{otherwise.} \end{cases}$$

We'll then only focus on the heights of our walkers. Again, we can push to a quotient Markov chain, using the fact that representatives in each height class have the same probability of moving to a representative in a different heigh class. We can then focus on the chain $\{0, 1, \ldots, k\}$ with transition probabilities nearly the same as above, except the probability of moving down from 0 is $1/2$. We set the coupling so that the walkers will move in the same direction with respect to height, that is they either move up, down, or stay together, with the caveat that at the root node the walker may sometimes move down while the other walker stays in place. We notice that once the walkers coalesce, they stay coalesced.

Claim 3.3. If $h(X_0) \leq h(Y_0)$, then we have $h(X_t) \leq h(Y_t)$ for all $t \geq 0$.

Proof. Notice that based on the coupling, if $h(Y_t) = h(Y_{t-1}) \pm 1$ or $h(Y_t) = h(Y_{t-1})$, then $h(X_t) = h(X_{t-1}) = \pm 1$ or $h(X_t) = h(X_{t-1})$ as well, with the exception at $h(X_{t-1}) = 0$. Here, we sometimes get $h(X_t) = h(X_{t-1}) + 1$ while $h(Y_t) = h(Y_{t-1}) \pm 1$ or $h(Y_t) = h(Y_{t-1})$. We will couple so that if $h(Y_t)$ increases by 1 then $h(X_t)$ will stay at 0, if $h(Y_t)$ stays the same then $h(X_t)$ will decrease by 1, and otherwise $h(X_t)$ will stay at 0. In such a case, we have coupled the walkers so that they either coalesce or $h(Y_t) > h(X_t)$. Therefore, we get Claim 3.3 holds, since once they coalesce they stay coalesced and otherwise we follow the same procedure again. ∎

As a result of the prior claim, it's sufficient to find out when $h(Y_s) = 0$ to determine when they coalesce. We then repeat the recursive argument. Let $f_i = \mathbf{E}(\tau \mid X_0 = i\}$. Then $f_0 = 0$,

$$f_j = \frac{1}{6}(1 + f_{j-1}) + \frac{1}{6}(1 + f_{j+1}) + \frac{2}{3}(1 + f_j)$$

for $0 < j < k$, and

$$f_k = \frac{1}{6}(1 + f_{k-1}) + \frac{5}{6}(1 + f_k).$$

Claim 3.4. We have

$$f_j = 3j + \frac{j}{j+1} f_{j+1}.$$

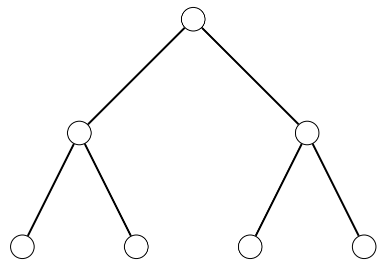

FIGURE 9. Binary tree of depth $k = 2$.

Proof. Again, we see that the base case holds. Assume it holds for $0 < j < k - 1$. Then we have

$$f_{j+1} = \frac{1}{6}(1 + f_j) + \frac{1}{6}(1 + f_{j+2}) + \frac{2}{3}(1 + f_{j+1}).$$

Substituting in appropriate values and solving this gives us

$$f_{j+1} = 2(j+1) + \frac{j+1}{j+2} f_{j+2}$$

as desired. ■

Claim 3.5. We have $f_i < f_{i+1}$ for all $0 \leq i < k$.

Proof. It is a result of Claim 3.4 and the construction. ■

We see that after substituting in the appropriate values from the above claim, we have $f_k = 3k(k+1)$. Combining our claims grants us

$$\max_{x,y \in \Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq 3k(k+1),$$

resulting in

$$t_{\text{mix}}(\epsilon) \leq \frac{3k(k+1)}{\epsilon}.$$

■

We now explore the binary tree with depth k. The binary tree is generated by attaching two vertices to every vertex of degree one; an example can be seen in Figure 9. We then explore the mixing time of this.

Proposition 3.6. For the lazy random walk on the binary tree of depth n, we have

$$\max_{x,y \in \Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq 6 \cdot 2^n.$$

Proof. The result can be found in Example 5.3.4 in [3]. We give the outline of the proof here. Let h again be the distance of a node from the root node; that is, $h(v) = \rho(v, v_*)$. Again, we can push to the quotient Markov chain using the height equivalence class, since transition probabilities between representatives are coherent. Thus, we can instead just consider the Markov chain on this sequence of numbers instead of the actual chain, with transition probabilities derived from the lazy random walk. Then we couple based on the distance from the root; that is, if one walker goes down, then the other walker goes down, and likewise for moving up. Doing so gives us the upper bound of

$$\max_{x,y \in \Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq 6 \cdot 2^n.$$

This matches our intuition, since this should be much larger than the result we found in Proposition 3.2. ■

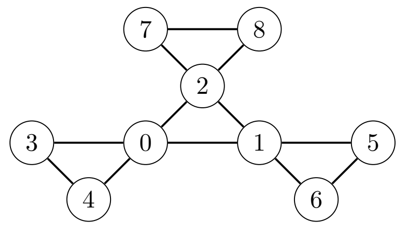

FIGURE 10. Expansion gluing $n = 3$, $k = 2$.

However, the coupling and argument didn't actually use the fact that there were only two paths leading out of the root node. Suppose we glued another copy of the binary tree of length $k-1$ to the root node of the binary tree of length k. Then similar to Proposition 3.1, we see that the mixing time is still the same.

We now introduce an iterated process for gluing copies of K_3 for Proposition 3.7. Throughout, we define an **open vertex** to be vertices whose degree is 2. At $k = 1$, we simply have the complete graph itself. At $k = 2$, we glue a copy of the graph on each vertex. At $k = 3$, we glue more copies along each open vertex. One can also think of gluing these copies along the boundary of our new graph. An example of this gluing procedure can be found in Figure 10.

One can then ask how we should expect the bound of the mixing time of the modified lazy random walk to behave as we increase k. We use the transition matrix introduced in Section 2; that is, we have

$$P(x,y) = \begin{cases} \frac{1}{8} & \text{if } y \in N(x), \\ \frac{1}{8}\big(4 - |N(x)|\big) + \frac{1}{2} & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Recall again that we do this so that the stationary distribution is uniform.

Proposition 3.7. For the above scenario, we have that for $k > 1$

$$t_{\mathrm{mix}}(\epsilon) \leq \frac{104}{3\epsilon}2^k - \frac{16}{\epsilon}k - \frac{100}{3\epsilon}$$

Proof. Let our random walkers start anywhere on the graph. We couple first based on the distance away from the first triangle. We change our height function to be

$$h(v) = \min_{x \in \{0,1,2\}} \{\rho(x,v)\}$$

where $0, 1, 2$ will be the vertices on the origin, or center, triangle, as seen in Figure 10. We have then transformed our random walk on the large graph to a random walk on $\{0, 1, \ldots, k\}$ such that when $h(X_t) = h(Y_t)$, we have $h(X_s) = h(Y_s)$ for all $s \geq t$. Imagine this set up on a line, with the leftmost position being the origin triangle and the rightmost triangle being the boundary. Then the probability of going left, or going towards the origin triangle, is $1/8$, the probability of going right, or away from the origin triangle, is $1/4$, and the probability of staying in place is $5/8$. We also have that trying to move left or right at the endpoints results in staying in place respectively. We will also couple our walks so that whenever X_t goes left, right, or stays in place, then Y_t also goes left, right, or stays in place respectively. It is sufficient then to find the expected amount of time it takes the walker with the highest height to hit 0. Without loss of generality, take this walker to be $\{Y_t\}$.

(0,1)
(0,0)
(1,1)
(2,2)
(3,5)
(4,6)

FIGURE 11. Locations and destinations of $(X_t = 0, Y_t = 1)$, with top row being the initial states and remaining being the coupled moves.

Let τ be the amount of time it takes for a walker to hit 0. We set $f_j = \mathbf{E}(\tau \mid X_0 = j)$. We have $f_0 = 0$,

$$f_k = \frac{7}{8}(1+f_k) + \frac{1}{8}(1+f_{k-1}),$$

and

$$f_j = \frac{5}{8}(1+f_j) + \frac{1}{4}(1+f_{j+1}) + \frac{1}{8}(1+f_{j-1})$$

for $0 < j < k$.

Claim 3.8. We have

$$f_j = \frac{8(2^{j+1}-j-2)}{2^{j+1}-1} + \frac{2^{j+1}-2}{2^{j+1}-1}f_{j+1}.$$

Proof. Solving for f_1 grants us

$$f_1 = \frac{8}{3} + \frac{2}{3}f_2,$$

as desired. Assume Claim 3.8 holds for j, $0 < j < k-1$. Then we must show the inductive hypothesis holds for $j+1$. We have

$$f_j = \frac{8(2^{j+1}-j-2)}{2^{j+1}-1} + \frac{2^{j+1}-2}{2^{j+1}-1}f_{j+1}$$

and

$$f_{j+1} = \frac{5}{8}(1+f_{j_1}) + \frac{1}{4}(1+f_{j+2}) + \frac{1}{8}(1+f_j).$$

Substituting in appropriate values gives

$$f_{j+1} = \frac{8(2^{j+2}-(j+1)-2)}{2^{j+2}-1} + \frac{2^{j+2}-2}{2^{j+2}-1}f_{j+2}.$$

■

Claim 3.9. We have $f_i < f_{i+1}$ for all $0 \le i < k$.

Proof. This is a result of Claim 3.8 and the construction itself. ■

The prior claims give us

$$f_k = 2^{k+4} - 16 - 8k.$$

So the expected amount of time for the walkers to coalesce is bounded above by f_k.

Now, we create another coupling. The walkers move the same direction on their respective triangles until they reach the origin triangle; that is, once $\rho(X_t, Y_t) \le 1$. We then have that if they try to move to one of the other two vertices on the origin triangle or stay in place, then they coalesce. If they otherwise move backwards, then they move backwards identically. An example of this can be found in Figure 11, with reference to Figure 10. This then gives us a random walk on the graph of their distance $\rho(X_t, Y_t) = \{0, 1, 3, 5, \ldots, 2k+1\}$, with absorption at 0, a 3/4 chance of going to 0 from 1, a 1/4 chance of going from 1 to 3, and the rest of the chain remains identical to the quotient chain mentioned prior. To make things easier, we'll rewrite this instead as $\{0, 1, 2, 3, \ldots, k, k+1\}$ (the k here is important, and as a result we cannot use a different variable here in its stead). We then want to find the expected amount of time until they coalesce at 0. Again, we set up a series of equations with $f_j = \mathbf{E}(\tau \mid X_0 = j)$, and we see $f_0 = 0$,

$$f_1 = \frac{3}{4} + \frac{1}{4}(1 + f_2),$$

$$f_{k+1} = \frac{7}{8}(1 + f_{k+1}) + \frac{1}{8}(1 + f_k),$$

and

$$f_j = \frac{5}{8}(1 + f_j) + \frac{1}{4}(1 + f_{j+1}) + \frac{1}{8}(1 + f_{j-1}),$$

for $1 < j < m + 1$.

Claim 3.10. We have

$$f_j = \frac{56 \cdot 2^{j-1} - 24j - 28}{7 \cdot 2^{j-1} - 3} + \left(\frac{7(2^{j-1}) - 6}{7(2^{j-1}) - 3}\right) f_{j+1}$$

for $0 < j < k$.

Proof. We proceed by induction again. We see that substituting in $j = 1$ gives us

$$f_1 = 1 + \frac{1}{4} f_2,$$

as required. We now assume Claim 3.10 holds for j, and show it holds for $j + 1$. We have

$$f_j = \frac{56 \cdot 2^{j-1} - 24j - 28}{7 \cdot 2^{j-1} - 3} + \left(\frac{7(2^{j-1}) - 6}{7(2^{j-1}) - 3}\right) f_{j+1}$$

and

$$f_{j+1} = \frac{5}{8}(1 + f_{j+1}) + \frac{1}{4}(1 + f_{j+2}) + \frac{1}{8}(1 + f_j).$$

Substituting in the appropriate values and solving for f_{j+1}, we have

$$f_{j+1} = \frac{56 \cdot 2^j - 24(j+1) - 28}{7 \cdot 2^j - 3} + \left(\frac{7(2^j) - 6}{7(2^j) - 3}\right) f_{j+2}$$

as desired. ■

Claim 3.11. We again get $f_j < f_{j+1}$ for $0 \leq j < k + 1$.

Proof. Again, a result from the prior claim and the construction. ■

We find that

$$f_{k+1} = \frac{112}{3} 2^{k-1} - 8k - \frac{52}{3}.$$

So the expected time for the two random walkers to coalesce once they've coupled using the prior couple is bounded by f_{k+1}. Hence, if τ is the amount of time for them to couple, we've found that

$$\max_{x,y \in \Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{104}{3} 2^k - 16k - \frac{100}{3},$$

where k is the number of times we've iterated the gluing procedure. We get then

$$t_{\text{mix}}(\epsilon) \leq \frac{104}{3\epsilon} 2^k - \frac{16}{\epsilon} k - \frac{100}{3\epsilon}$$

as desired. ■

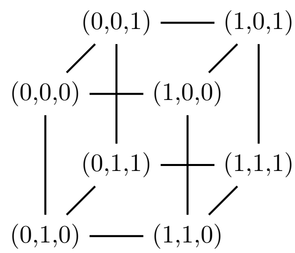

FIGURE 12. 3-dimensional hypercube.

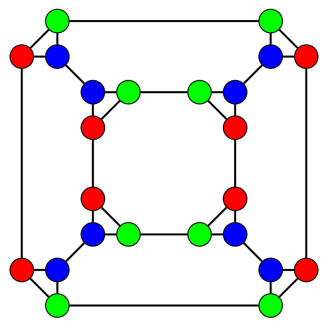

FIGURE 13. 3-dimensional triangulated hypercube.

4. Mixing Times on Nice Classes of 3-Regular Graphs

Consider the space of binary strings of length n; that is, let

$$\Omega = \{0,1\}^n = \{v = (v_1, \dots, v_n) \ : \ v_i \in \{0,1\}\}.$$

Definition 4.1. We define the **Hamming weight** on this space to be a function $H : \Omega \to \{0, \dots, n\}$ such that

$$H(v) = \sum_{i=1}^{n} v_i.$$

In other words, we sum the components of this vector.

Definition 4.2. We form the **n-dimensional Hypercube** by taking the space Ω to be the set of vertices for our graph, and we connect an edge between $v, w \in \Omega$ if

$$|H(v) - H(w)| = 1.$$

We have that the 3-dimensional Hypercube forms a very nice 3-regular graph, in that it is **vertex transitive**.

Definition 4.3. A graph G is **vertex transitive** if, for any two vertices v_1, v_2, there is some map $f : G \to G$ preserving edge-vertex connectivity (also referred to as a **graph automorphism**) with

$$f(v_1) = v_2.$$

Remark 4.4. All of the graphs mentioned in this section will be vertex transitive. This may be important (see the Question 4.23), however we do not really use this property in finding any of the mixing times themselves. When referring to "nice class of graphs," we mean a family of vertex transitive 3-regular graphs which are easily describable.

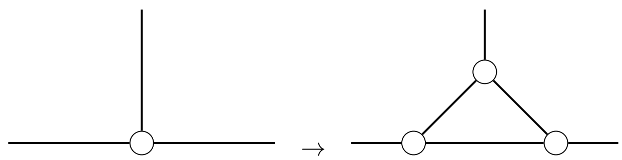

FIGURE 14. Triangulating a vertex on a 3-regular graph

An example of this graph can be seen in Figure 12. The mixing time of the lazy random walk on this graph is well understood to be

$$t_{\text{mix}} \leq n\log(n) - n\log(\epsilon)$$

as seen in Example 5.3.1 in [3]. We would like to understand the behavior of the modified lazy random walk on the **triangulated** 3-dimensional hypercube. By triangulated, we mean replacing each vertex with a copy of K_3 and attaching corresponding edges. An example can be seen of the process can be seen in Figures 14 and the triangulated version of the 3-dimensional hypercube can be seen in 13.

Proposition 4.5. For the lazy random walk on the triangulated 3-dimensional hypercube, we have

$$t_{\text{mix}}(\epsilon) \leq \frac{42}{\epsilon}.$$

Proof. Recall that the lazy random walk on this graph will be

$$P(x,y) = \begin{cases} \frac{1}{2} & \text{if } x = y, \\ \frac{1}{6} & \text{if } y \in N(x), \\ 0 \text{ otherwise.} \end{cases}$$

Notice that the vertices on our triangle correspond to dimensions, as can be seen in Figure 13. That is, we have that the blue vertices correspond to moving in and out, the red vertices correspond to moving up and down, and the green vertices correspond to moving left and right in Figure 13. Assume that our walkers do not match in any dimensions at their start. Then the expected amount of time until they match in some dimension is a geometric random variable with a probability of success being 1/6. The expected time for this is then 6. Now, our walkers will move together in the dimension that they have matched up in, and will move independently on either two dimensions. We now want to measure the expected amount of time until our walkers match in another dimension. We now have three states to consider; if our walkers are on the vertex color they have matched in, if they are on a new color they have not yet matched in, and if they have coalesced in a new color or dimension. We denote these by 2, 1, and 0 respectively. Doing so now transforms our Markov chain to a walk on $\{0, 1, 2\}$. Let τ' be the amount of time it takes until our walker on this new chain reaches 0. Then we set $f_i = \mathbf{E}(\tau' \mid X_0 = i)$, giving us $f_0 = 0$,

$$f_1 = \frac{1}{6} + \frac{2}{3}(1 + f_1) + \frac{1}{6}(1 + f_2),$$

and

$$f_2 = \frac{2}{3}(1 + f_2) + \frac{1}{3}(1 + f_1).$$

Solving this sequence of equations gives $f_2 = 12$ as our upper bound for this portion. Now with our walkers matching in 2 dimensions, we want to see how long it takes until they match in the third, and therefore coalesce. We again have the three options we had before; our walkers are on a vertex in which they already match

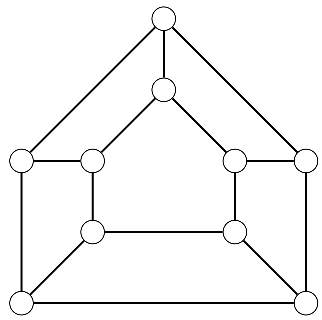

FIGURE 15. Prism graph of size 10

in dimension, they are on a vertex which they do not match in dimension, or they have coalesced in this final dimension. Again, we represent these states as 2, 1, and 0. We have that our Markov chain is a walk on $\{0, 1, 2\}$, with 0 as the absorbing state. Using the same notation as before, we have $f_0 = 0$,

$$f_1 = \frac{1}{6} + \frac{1}{3}(1 + f_2) + \frac{1}{2}(1 + f_1),$$

and

$$f_2 = \frac{5}{6}(1 + f_2) + \frac{1}{6}(1 + f_1).$$

Solving this series of equations gives us $f_2 = 24$, and so we have an upper bound of 24. If τ is the amount of time until the walkers coalesce, then we have a bound of

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq 6 + 12 + 24 = 42$$

as desired. ■

This then begs the question of whether or not there is some relation between the mixing time of the modified random walk on the triangulated graph and the mixing time of the modified walk on the 3-regular graphs. To explore this, we first study the **prism graph**.

Definition 4.6. The **prism graph** on $2n$ vertices is constructed by taking two cycles of length n and attaching edges between corresponding vertices. Notationally, we have $G = (V, E)$ is the prism graph if

$$V = \{(a, b) \mid a \in \{0, 1\}, b \in \{0, \dots, n-1\}\},$$

$$E = \{((a, b), (c, d)) \mid a = c, b = d \pm 1 (\text{mod } n)\}.$$

An example of a prism graph can be seen in Figures 12 and 15.

Proposition 4.7. For the lazy random walk on the prism graph of size n, we have

$$t_{\text{mix}} \leq \frac{3n^2}{16\epsilon} + \frac{6}{\epsilon}$$

Proof. We first couple based on whether our walkers are both on the inner or outer cycles. Flipping a coin to decide which walker moves and letting them move wherever possible, we see that this it is a geometric random variable until our walkers match in cycle. Since there is a probability of 1/6 of the walker moving into the same cycle at each step, we have that the expected amount of time is 6. It is now simply a random walk on the cycle with a higher chance of staying in place

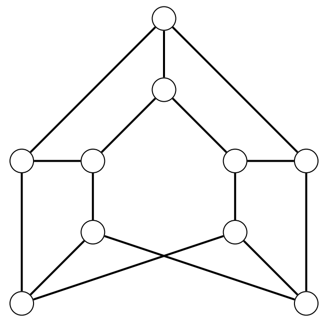

FIGURE 16. Möbius ladder graph of size 10

(2/3 chance of staying in place, and 1/6 chance of going left or right). Running through the process, we have

$$f_k = 3k + \frac{k}{k+1} f_{k+1}$$

leading us to have

$$f_k = 3k\left(\frac{n}{2} - k\right).$$

Maximizing this gives

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = y, Y_0 = y) \le \frac{3}{16}n^2 + 6$$

as desired. ∎

This argument lets us also find the mixing time of the **Möbius ladder graph**.

Definition 4.8. The **Möbius ladder graph** is formed by doing the same construction as the prism graph, except there's a twist at the bottom. Notationally, we have $G = (V, E)$ is the Möbius ladder graph of size $2n$ if

$$V = \{(a, b) \mid a \in \{0, 1\}, b \in \{0, \dots, n-1\}\},$$
$$E = \{((a, b), (c, d)) \mid a = b \text{ and } b = d \pm 1 (\text{mod } n), b \in \{1, \dots, n-2\}$$
$$\text{or } a = 1 - b \text{ and } b = d + 1 (\text{mod } n) \text{ if } d = n-1, b = d - 1 (\text{mod } n) \text{ if } d = 0\}.$$

An example can be seen in Figure 16. Since the Möbius ladder graph can be unwound, as seen in Figure 17, we could alternatively identify this as

$$V = \{0, \dots, 2n-1\}$$
$$E = \{(a, b) \mid a = b \pm 1 (\text{mod } 2n) \text{ or } a = b \pm n (\text{mod } 2n)\}.$$

Corollary 4.9. For the modified lazy random walk on the Möbius ladder graph, we have that the mixing time is

$$t_{\text{mix}} \le \frac{3n^2}{16\epsilon} + \frac{6}{\epsilon}.$$

Proof. Notice that the Möbius ladder graph can be unwound to get the circulant graph, as seen in Figure 17. We identify two vertices across from one another to be in the same class, that is, vertices $v_1, v_2 \in V$ are in the same class if $v_1 = v_2 + n/2 (\text{mod } n)$. This pushes our Markov chain to its quotient Markov chain. We see we have a cycle on $n/2$, with modified probability of staying in place as in Proposition 4.7. Moreover, once coupled in class, we just need to couple in position itself. Flip a coin and move the walker; if it jumps across, we have the walkers coalesce and we win. Otherwise, move the other walkers so that they are still in the

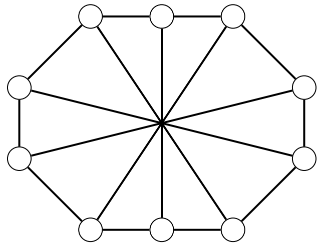

FIGURE 17. Unwound Möbius ladder graph of size 10.

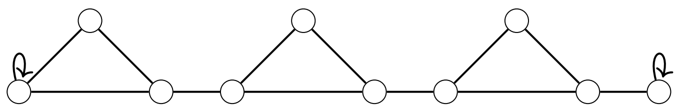

FIGURE 18. Triangulated cycle distance. Notice that we went from 3 vertices to $3(3) + 1 = 10$ vertices.

same class. Then the probability of the walkers coalescing is a geometric random variable with probability $1/6$, and so we get the same constant factor of 6 as an upper bound for this latter bound. Summing our two results together gives the result. ■

Proposition 4.10. For the lazy random walk on the triangulated prism graph of size n, we have

$$t_{\text{mix}}(\epsilon) \leq \frac{15n^2}{16\epsilon} + \frac{87}{5\epsilon}.$$

Proof. We again want to start by coupling based on which cycle (either inner or outer) we are in. Letting the walkers move independently of one another, we wait until they occupy the same location relative to their triangles. This will take 6 steps, since this is just a geometric random variable. Couple the walkers so that they now move the same direction on their relative triangles. We now wait until they reach the vertex corresponding to moving either inwards or outwards, and then couple it so that there's a $1/3$ chance they move to the same level, $1/3$ chance they stay in place, and $1/3$ chance they move backwards together. On the remaining two nodes, there is a $1/6$ chance they move to the top node and a $5/6$ chance they stay on the base of the triangle. Solving the equations gives us that this will take 15 steps.

Now that we have they are in the same cycle, we have that they will stay in the same cycle. We now focus on the lazy random walk on the triangulated cycle, with the addendum that on the vertices of degree 2 (the top vertices of the triangles) we have a $2/3$ chance of staying in place. We'll couple so that they move to the top vertex together, move independent of one another on the bottom vertices (flip a coin and move either left or right), and otherwise move to opposite locations on the base from the top vertex. Using the same philosophy as with the normal cycle, we instead look at the Markov chain constructed by the clockwise distances between the vertices, with absorption at 0 and $3n + 1$ (here, $3n + 1$ since there are now $3n$ vertices and we add an extra vertex to denote coalescing). An example of this can be seen in Figure 18.

Claim 4.11. Let g_i be the expected value of being absorbed starting at the right most vertex in the i^{th} triangle, h_i the left most vertex on the i^{th} triangle, and f_i

on the top most vertex on the $i+1^{\text{th}}$ triangle. Then we have

$$g_i = \frac{18+45(i-1)}{5} + \frac{5i-3}{5i}h_{i+1}.$$

Proof. This can be shown using the usual inductive argument. In the case of $i=1$, we get

$$g_1 = \frac{18}{5} + \frac{2}{5}h_2$$

by simply substituting in the appropriate values. Assuming it holds for i, we simply need to substitute in appropriate values on this chain and we should get the result. So we have

$$g_i = \frac{18+45(i-1)}{5} + \frac{5i-3}{5i}h_i,$$

$$h_i = \frac{1}{6}(1+g_i) + \frac{1}{6}(1+f_i) + \frac{1}{6}(1+g_{i+1}) + \frac{1}{2}(1+h_i),$$

and

$$f_i = \frac{1}{6}(1+h_i) + \frac{1}{6}(1+g_{i+1}) + \frac{2}{3}(1+f_i).$$

Solving all of these variables and then solving for

$$g_{i+1} = \frac{1}{6}(1+h_i) + \frac{1}{6}(1+f_i) + \frac{1}{6}(1+g_{i+2}) + \frac{1}{2}(1+g_{i+1})$$

gives us

$$g_{i+1} = \frac{18+45i}{5} + \frac{5i+2}{5i+5}h_{i+1}.$$

as desired. ∎

Claim 4.12. Letting $k = n/2$, we have

$$g_l = \frac{3}{5}(5k-5l+3)(5l-3).$$

Proof. This holds for the base case $l=k$. Assume it holds for i. We must show it holds for $i-1$. Going through the motions, we have

$$f_i = \frac{1}{6}(1+g_i) + \frac{1}{6}(1+h_i) + \frac{2}{3}(1+f_i)$$

which solving gives us

$$f_i = \frac{3}{10} + \frac{15}{2}ki - \frac{9}{2}k - \frac{15}{2}i^2 + 9i + \frac{1}{2}h_i.$$

Solving

$$h_i = \frac{1}{6}(1+g_{i-1}) + \frac{1}{6}(1+f_i) + \frac{1}{6}(1+g_i) + \frac{1}{2}h_i$$

gives us

$$h_i = \frac{9}{25} + 9ki - \frac{27}{5}k - 9i^2 + \frac{54}{5}i + \frac{2}{5}g_{i-1}.$$

Finally, using the prior claim, we have

$$g_{i-1} = \frac{18+45\left((i-1)-1\right)}{5} + \frac{5(i-1)-3}{5(i-1)}h_i.$$

Substituting in the h_i we just found, we get

$$g_{i-1} = -\frac{3}{5}(5i-5k-8)(5i-8),$$

which satisfies the induction hypothesis. ∎

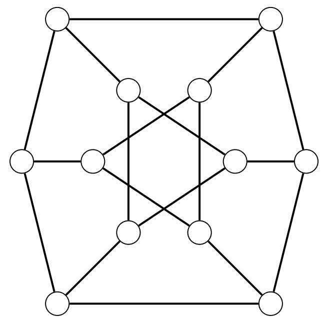

FIGURE 19. The generalized Petersen graph GP$(6, 2)$.

This then also gives us

$$h_l = 15(l-1)(k-l+1)$$

and

$$f_l = -\frac{36}{5} + 15\,kl - 12\,k - 15\,l^2 + 24l$$

for free by substitution. The top most node will be furthest away, so maximizing f_l with respect to l gives

$$\frac{15}{16}n^2 + \frac{12}{5}.$$

Adding in the constant of 21 we found earlier gives us

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{15}{16}n^2 + \frac{87}{5}.$$

■

Corollary 4.13. For the modified lazy random walk on the triangulated Möbius ladder graph, we have that the mixing time is

$$t_{\text{mix}}(\epsilon) \leq \frac{15n^2}{16\epsilon} + 875\epsilon.$$

Proof. The argument is analogous to the argument found in Corollary 4.9. ■

We then can also explore another nice class of 3-regular graphs called **generalized Petersen graphs**, denoted by GP(n, k).

Definition 4.14. The generalized Petersen graph is constructed by having a cycle graph of size n on the outside connected to the star polygon of size n with k vertices between a vertex and its neighbor. Notice that we require $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. Notationally, we have again that a graph $G = (V, E)$ is the generalized Petersen graph $G(n, k)$ if

$$V = \{(a, b) \mid a \in \{0, 1\}, b \in \{0, \ldots, n-1\}\},$$

$$E = \{((a, b), (c, d)) \mid a \neq c, b = d \text{ or}$$

$$a = c = 1, b = d \pm 1 (\text{mod } n) \text{ or } a = c = 0, b = d \pm k (\text{mod } n)\}.$$

Prism graphs are an example of $GP(n, 1)$ for example. Another example can be seen in Figure 19. We can use a similar sort of coupling procedure as we used with the prism and Möbius graphs to find a bound for GP(n, k).

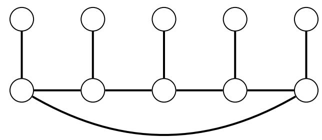

FIGURE 20. The cycle with extra nodes.

Proposition 4.15. For the lazy random walk on $\mathrm{GP}(n,k)$, we have that the mixing time is

$$t_{\mathrm{mix}} \leq \frac{3d^2}{2\epsilon} + \frac{3}{2\epsilon}\left(\frac{n}{d}\right)^2 + \frac{18}{\epsilon},$$

where $d = n/\gcd(n,k)$.

Proof. We first couple our walkers so that they have the same height; i.e., they are either on the inner cycles or on the outer cycle. Letting the walkers move independently of one another, we find that the probability of them coalescing with regards to inner or outer cycle, or height, is a geometric random variable with probability $1/6$. The expected value is then 6. Once the walkers share the same height, they will always share the same height. The number of different cycles on the inner part is $d = n/\gcd(n,k)$, since we're looking at the order of k in $\mathbb{Z}/n\mathbb{Z}$. We'll have our walkers move together on the inner part (so they move left, right, or outward together), and on the outer part they'll move independently of one another. Then we see that this gives us the quotient Markov chain on the cycle n/d with extra nodes coming outward. An example of this can be seen in Figure 20. We can use the same philosophy as on the normal cycle; shifting this to a Markov chain on $\{0,\dots,n/d\}$ with absorbing 0 and n/d being absorbing states. Focusing only on the nodes on the 'inside' portion, we set up a series of equations again.

Claim 4.16. If τ is the amount of time it takes to be absorbed, letting $f_j = \mathbf{E}(\tau \mid X_0 = j)$, we have

$$f_j = 6j + \frac{j}{j+1} f_{j+1},$$

$1 < j < n/d - 1$.

Proof. Notice that we have

$$f_1 = \frac{1}{6}(1+0) + \frac{1}{6}(1+f_2) + \frac{1}{6}(1+g_1) + \frac{1}{2}(1+f_1),$$

where g_i here denotes the outer nodes. For g_i, we have

$$g_i = \frac{5}{6}(1+g_i) + \frac{1}{6}(1+f_i),$$

and solving this gives

$$g_i = 6 + f_i.$$

Substituting in all the values gives

$$f_1 = 6 + \frac{1}{2} f_2,$$

so we have Claim 4.16 holds for the base case. Now, assume the inductive hypothesis holds for j. We show it holds for $j+1$. We set up

$$f_{j+1} = \frac{1}{6}(1+f_j) + \frac{1}{6}(1+g_j) + \frac{1}{6}(1+f_{j+2}) + \frac{1}{2}(1+f_j).$$

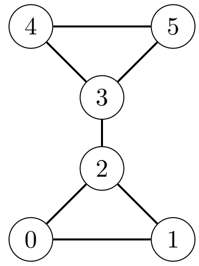

Figure 21. Triangles in the Cycle

Substituting in values and solving gives

$$f_{j+1} = 6(j+1) + \frac{j+1}{j+2} f_{j+2}$$

as desired. ■

Claim 4.17. We have

$$f_j = 6j\left(\frac{n}{d} - j\right).$$

Proof. From the prior claim, we have

$$f_{n/d-1} = 6\left(\frac{n}{d} - 1\right).$$

Assume Claim 4.17 holds by induction for i, and we want to show it holds for $i-1$. Then we have

$$f_i = 6i\left(\frac{n}{d} - i\right)$$

and

$$f_{i-1} = 6(i-1) + \frac{i-1}{i} f_i.$$

Substituting in the values gives us

$$f_{i-1} = 6(i-1)\left(\frac{n}{d} - i + 1\right)$$

as desired. ■

Maximizing this, and using the fact that $g_i > f_i$ for all i, we have an upper bound of $(3/2)(n/d)^2 + 6$. Now that the walkers are in the same inner cycle (since there are d inner cycles that we could possibly be in), we can couple with respect to the inner cycle, measuring how long it takes for the walkers to coalesce on their respective inner cycle. However, this is just the same process as in Claims 4.16 and 4.17, replacing n/d with d. Once we have this, we sum everything together to see that

$$d(t) \leq \frac{3}{2t}\left(\frac{n}{d}\right)^2 + \frac{3d^2}{2t} + \frac{18}{t}.$$

as desired. ■

Proposition 4.18. For the lazy random walk on the triangulated $\mathrm{GP}(n,k)$, we have that the mixing time is

$$t_{\text{mix}} \leq \frac{15d^2}{2\epsilon} + \frac{15}{2\epsilon}\left(\frac{n}{d}\right)^2 + \frac{9n}{d\epsilon} + \frac{9d}{\epsilon} + \frac{96}{\epsilon}.$$

Proof. We will walk through roughly the same steps as before. We first couple our walkers so they are in the same position in their relative triangle. Moving our walkers independently, this is a geometric random variable and so it takes on average 6 steps. We then want to couple our walkers so that they are in the same 'height' on their respective triangle duo. See Figure 21 for an example. We couple so that if our walkers move to the top of their respective triangles (2 or 3 in Figure 21), then they move together. Once they are at the top of their triangles, then they coalesce with probability 1/3 (one hops and the other stays in place), stay with probability 1/3, and move downward with probability 1/3. and once they are coupled in height then they stay coupled in height. Setting up a series of recursive functions, we find that this is bounded by 15. Once coupled with regards to height, we then want to couple based on which inner cycle we are in, as in the proof of Proposition 12.

This gives us a cycle of length n/d of triangles with triangles above it as well. An example of what each state looks like can be seen in Figure 21 again. We then would like to find the maximum expected amount of time it takes to coalesce on the cycle. Let the walkers move together everywhere except for the base of the triangle and left and right independently on the base of the triangle (that is, one stays in place while the other moves left or right). When moving downward from the top vertex of the base triangle, the walkers move to opposing spaces. We can then preform a similar procedure as in the proof of Proposition 11, with now the maximum expected time being at either of the top two vertices.

Claim 4.19. Take f_i to be the far right vertex on the base of the i^{th} triangle (for example, 1 in Figure 21) and h_i the left vertex on the base of the i^{th} triangle. Then we have

$$f_i = 18i + \frac{8+5(i-1)-3}{8+5(i-1)}h_{i+1}.$$

Proof. We proceed by induction. For the base case, we just plug in values to get

$$f_1 = 18 + \frac{5}{8}h_2.$$

Let k_i be the vertex above h_i and f_i (for example, 2 in Figure 21), a_i be the vertex above that (for example, see 3 in Figure 21), d_i and g_i the vertices on the top left and right respectively (for example, see 4 and 5 respectively in Figure 21). Now, we assume Claim 4.19 holds for i and show it holds for $i+1$. Using our series of equation, we have

$$\begin{aligned}
f_i &= 18i + \frac{8+5(i-1)-3}{8+5(i-1)}h_{i+1},\\
h_{i+1} &= \frac{1}{6}(1+f_{i+1}) + \frac{1}{6}(1+f_i) + \frac{1}{6}(1+k_{i+1}) + \frac{1}{2}(1+h_{i+1}),\\
k_{i+1} &= \frac{1}{6}(1+h_{i+1}) + \frac{1}{6}(1+f_{i+1}) + \frac{1}{6}(1+a_{i+1}) + \frac{1}{2}(1+k_{i+1}),\\
a_{i+1} &= \frac{1}{6}(1+k_{i+1}) + \frac{1}{6}(1+d_{i+1}) + \frac{1}{6}(1+g_{i+1}) + \frac{1}{2}(1+a_{i+1}),\\
d_{i+1} &= \frac{1}{6}(1+a_{i+1}) + \frac{1}{3}(1+g_{i+1}) + \frac{1}{2}(1+d_{i+1}),\\
g_{i+1} &= \frac{1}{6}(1+a_{i+1}) + \frac{1}{3}(1+d_{i+1}) + \frac{1}{2}(1+g_{i+1}),\\
f_{i+1} &= \frac{1}{6}(1+h_{i+1}) + \frac{1}{6}(1+k_{i+1}) + \frac{1}{6}(1+h_{i+2}) + \frac{1}{2}(1+f_{i+1}).
\end{aligned}$$

Solving and substituting in values gives us

$$f_{i+1} = 18(i+1) + \frac{5+5i}{8+5i}h_{i+2}$$

as desired. ■

Claim 4.20. Without loss of generality, take g_i be the top right vertex of the i^{th} triangle (for example, 5 in Figure 21), and take f_i to be the far right vertex on the base (for example, 1 in Figure 21). Then we have

$$g_i = \frac{198 + 30(i-1)}{5} + \frac{5i-1}{5i} f_i.$$

Proof. Manually going through, we see we have

$$g_1 = \frac{198}{5} + \frac{4}{5} f_1.$$

Using the proof from the prior claim, we see that

$$g_{i+1} = \frac{198 + 30(i)}{5} + \frac{5(i+1)-1}{5(i+1)} f_{i+1}$$

as desired. ■

Claim 4.21. We have

$$f_i = 6i\left(5\left(\frac{n}{d}\right) - 5i + 3\right).$$

Proof. Let $\alpha = n/d = \gcd(n,d)$ for notational simplicity. We have $h_{\alpha+1} = 0$, and so it follows that $f_\alpha = 18\alpha$, as desired from Claim 4.19. Now we check it holds for induction. Assume it holds for $i+1$, then we want to show it holds for i. Then using the values we found from the proof of Claim 4.19, we substitute in

$$f_{i+1} = 6(i+1)\left(5\alpha - 5(i+1) + 3\right)$$

and we see that

$$f_i = 6i(5\alpha - 5i + 3)$$

as desired. ■

Claim 4.22. We have

$$g_i = -30i^2 + (30\alpha + 30)\, i - 6\alpha + 30$$

Proof. A consequence of Claims 4.20 and 4.21. ■

Maximizing g_i with respect to this gives us

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{15}{2}\left(\frac{n}{d}\right)^2 + 9\frac{n}{d} + \frac{75}{2}.$$

As in Proposition 12, we repeat this process for the inner triangles, with coalescing here now meaning our walkers coalesce in the larger triangle. Repeating this procedure now gives

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{15}{2}d^2 + 9d^2 + \frac{75}{2}.$$

We then sum these together along with our constant value to get our result. ■

In the prism graph, Möbius ladder graph, and generalized Petersen graph, we see that (fixing k) the triangulated version is asymptotically 5 times larger than the non-triangulated version of the graph. This then leads us to a few questions.

Question 4.23. For vertex transitive 3-regular graphs, is the mixing time of the triangulated 3-regular graph at most asymptotically 5 times larger than the upper bound of the mixing time of the non-triangulated version?

Question 4.24. For all 3-regular graphs, is the mixing time of the triangulated 3-regular graph always asymptotically 5 times larger than the upper bound of the mixing time of the non-triangulated version?

Question 4.25. Can we find similar results for the lower bounds on these mixing times?

Question 4.26. Are these bounds tight?

Further research on this could explore these questions. Propositions 4.7, 4.10, 4.15, and 4.18 all give some evidence towards Question 4.1. Being able to classify graphs like the Heawood and Franklin graph may provide further evidence or counterexamples for Question 4.1. For Questions 4.2 and 4.3, one may want to explore 3-regular graphs with large bottlenecks. It seems like vertex transitivity is a required property for this to work based on how the couplings in the prior examples worked, so explicitly finding a family of 3-regular graphs which are not vertex transitive and determining the asymptotics of the mixing time for the non-triangulated and triangulated version may provide counter examples for Question 4.2. Finally, methods other than coupling may shed some light on Question 4.4.

5. Appendix

Throughout, we mention things like Gambler's ruin and the mixing time on the cycle of length n. While these facts can be found in most textbooks on Markov chains and mixing times (see [2], [3], [4]), we place them here for convenience.

Proposition 5.1 (Fair Gambler's Ruin)**.** If a Gambler is betting 1 dollar on flips of a fair coin, and must leave the game if they run out of money or reach n dollars, then we have

$$\mathbf{E}(\tau \mid X_0 = k) = k(n-k),$$

where τ is the amount of time it takes to reach 0 or n.

Proof. Let $f_i = \mathbf{E}(\tau \mid X_0 = i)$. Then we have $f_0 = 0 = f_n$, and

$$f_i = \frac{1}{2}(1 + f_{i-1}) + \frac{1}{2}(1 + f_{i+1}).$$

This leads us to the following claim.

Claim 5.2. We have

$$f_i = i + \frac{i}{i+1} f_{i+1}$$

for $0 < i < n$.

Proof. We proceed by induction. For the base case, we have

$$f_1 = \frac{1}{2}(1+0) + \frac{1}{2}(1+f_2)$$

and so simplifying gives us

$$f_1 = 1 + \frac{1}{2} f_2.$$

Assume it holds for i. Then we have

$$f_{i+1} = \frac{1}{2}(1+f_i) + \frac{1}{2}(1+f_{i+2}).$$

Substituting in f_i and solving for f_{i+1} gives

$$f_{i+1} = i + 1 + \frac{i+1}{i+2} f_{i+2}$$

as desired. ∎

We now need to establish the claim by induction. For the base case, we have

$$f_{n-1} = n - 1 + \frac{n-1}{n}(0) = n - 1 = (n-1)\big(n-(n-1)\big).$$

Assume it holds for k; that is,

$$f_k = k(n-k).$$

Then we have

$$f_{k-1} = k - 1 + \frac{k-1}{k} f_k.$$

By the prior claim, we get

$$\begin{aligned} f_{k-1} &= k - 1 + \frac{k-1}{k}\big(k(n-k)\big) = k - 1 + (n-k)(k-1) \\ &= \big(1 + (n-k)\big)(k-1) = (k-1)\big(n-(k-1)\big) \end{aligned}$$

as desired. So the inductive hypothesis holds, and we get

$$f_k = k(n-k)$$

for all $0 < k < n$ as desired. In particular, maximizing this gives

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{n^2}{4}.$$

■

Proposition 5.3 (Mixing Time of the Cycle)**.** We have that the mixing time on the cycle $\mathbb{Z}/n\mathbb{Z}$ is bounded above by

$$t_{\text{mix}}(\epsilon) \leq \frac{n^2}{4\epsilon}.$$

Proof. Our coupling procedure is as follows: flip a fair coin to determine which walker we move. Then, flip another fair coin to determine whether the walker moves left or right. We measure the clockwise distance between the two walkers after each step. The coupling procedure then shifts us to a walk on the chain $\{0\dots,n\}$, with 0 and n being absorbing states and transition probabilities being 1/2. We can then use Proposition 5.1 to determine that

$$\max_{x,y\in\Omega} \mathbf{E}(\tau \mid X_0 = x, Y_0 = y) \leq \frac{n^2}{4},$$

and so using Theorem 1.20 we get

$$t_{\text{mix}} \leq \frac{n^2}{4\epsilon}.$$

■

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CUT-PASTE OPERATIONS AND BORDISM OF MANIFOLDS IN AN EQUIVARIANT SETTING

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CONTENTS

ABSTRACT. This paper expands on previous results of equivariant and non-equivariant SK. We present oriented analogs of some equivariant results presented by Komiya, Hara, and Waner & Wu. In addition to building on these results, we introduce a new exact sequence that connects non-equivariant SKK and SK in both the oriented and non-oriented cases. We then extend this result to an equivariant setting to produce an equivariant version of the exact sequence. We also incorporate the notion of SKV as introduced by Prevot, to manage torsion induced by the Korvaire Semicharacteristic.

1. INTRODUCTION

The purpose of this project is to generalize and expand upon previous results about the relations between equivariant SK operations and bordism relations for both oriented and non-oriented cases. In Section 2, the classical non-equivariant results of [KKNO73] are presented, along with the nonequivariant version of main result in Theorem 2.3. In Section 3, requisite knowledge of equivariance and representation theory is introduced, while in Section 4 equivariant *nonoriented* analogs of the classical results are presented. Section 5 presents equivariant *oriented* analogs of the classical results, in addition to our main results. Our main result produces a new exact sequence connecting SKK and SK previously unseen in the literature. Theorem 2.3 and Theorem 2.4 introduce this sequence in the non-equivariant setting, while Theorem 5.2 and Theorem 5.3 generalize the non-equivariant theorems to an equivariant setting.

2. NON-EQUIVARIANT SK OPERATIONS AND BORDISM

In this section, we will be presenting classical results classifying manifolds non-equivariantly. Let $\mathcal{M}_n$ be the collection of n-dimensional closed (oriented) manifolds classified up to (orientation preserving) diffeomorphism. Let $\mathcal{M}_n^O$ denote the non-oriented analog. By giving this collection an addition operation defined by disjoint union and additive identity , the empty manifold, we see that it clearly becomes a commutative monoid. Now, we would like to introduce the notion of cut-paste or SK (Schneiden-Kleben) relations.

Definition 2.1. *Let N, N' be (oriented) n-manifolds with (oriented) boundaries ∂N and $\partial N'$ respectively and let $\varphi, \psi : \partial N \to \partial N'$ be (orientation preserving) diffeomorphisms. Now let M be an (oriented) n-manifold such that $M = N \cup_\varphi -N'$, (where $-N'$ is N' with reversed orientation). A* **cut-paste (SK) operation** *on M produces a new (oriented) manifold M' by the following procedure: Cut M along ∂N to produce $N + (-N)'$ and then paste by $\psi : \partial N \to \partial N'$ to produce the new manifold $M' = N \cup_\psi -N'$.*

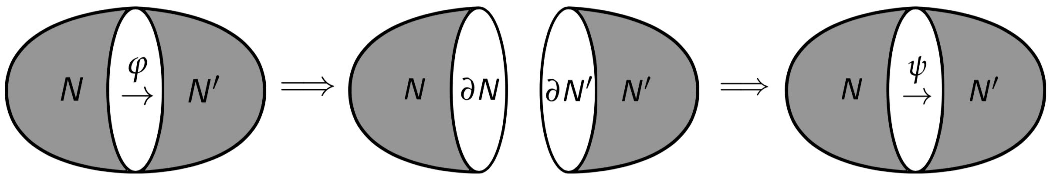

FIGURE 1. SK Operation

These operations define a relation that allows us to factor $\mathcal{M}_n$ into equivalence classes as follows:

Definition 2.2. *M, M' are* **SK-equivalent** *if there exist some n-manifold W such that $M' + W$ can be obtained from $M + W$ by SK operations.*

Definition 2.3. *We now produce the group of n-dimensional oriented SK-equivalent manifolds by taking $\mathcal{M}_n$ and quotienting by SK-equivalence. We then take the Grothendieck completion to produce the group $\mathbf{SK_n}$. An analogous factoring in the non-oriented case can be done to produce $\mathbf{SK_n^O}$*

The constructed inverses of SK elements are mentioned later in subsection4.1, and their full description is given in [KKNO73, Pg. 57]. SK groups are of primary interest due to the study of SK invariants. Examples of SK invariants include Euler characteristic χ and signature τ. In general, they are defined as follows:

Definition 2.4. *Let G be a group and $\rho : \mathcal{M}_n \to G$ (alternatively $\rho : \mathcal{M}_n^O \to G$) be a monoid homomorphism. ρ is an* **SK-invariant** *if $\rho(M) = \rho(M')$ for M, M' SK-equivalent manifolds.*

Showing that Euler characteristic χ and signature τ are SK invariants follows from the additive properties of these functions. For χ, When we let $X = A \cup B$, we have the following additive property.

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

So by taking collared neighborhoods around N and N' which intersect at the gluing at ∂N, we see that

$$\chi(M) = \chi(N) + \chi(N') - \chi(\partial N) = \chi(M').$$

The signature being SK-invariant follows from the property of Novikov additivity which states that

$$\tau(M) = \tau(N) - \tau(N') = \tau(M').$$

In fact, [KKNO73, Corollary 1.4] states that all SK invariants are linear combinations of Euler characteristic and signature.

Now, an equivalent formulation of an SK-invariant ρ is that, when M and M' are SK equivalent, $\rho(M) - \rho(M') = 0$. We may now consider another class of homomorphisms that generalize SK-invariants, where instead of the difference being zero, it is some error term dependent only on the gluing maps. This introduces a new class of invariant defined as follows.

Definition 2.5. *An* **SKK-Invariant** *is a monoid homomorphism* $\lambda : \mathcal{M}_n \to G$ *(alternatively* $\lambda : \mathcal{M}_n^O \to G$*) where for* $M = N \cup_\varphi -N'$ *and* $M' = N \cup_\psi -N'$ *then*

$$\lambda(M) - \lambda(M') = \lambda(\varphi, \psi).$$

Such an invariant is called SK-controllable, (SK-Kontrollierbar) as it is controllable solely through the choice of boundary diffeomorphism, and not the manifolds with boundary N and N'. Notice that all SK-invariants are SKK-invariants with error term 0. We now present an example of an SKK-invariant which is not an SK-invariant. Recall the Euler Characteristic of n-dimensional manifold:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \text{rank}(H_i(X; \mathbb{Z})).$$

Definition 2.6. *The* **Kervaire Semicharacteristic** *is defined to be:*

$$k(M) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \text{rank}(H_i(X; \mathbb{Z})) \pmod 2.$$

This function only takes non-trivial value on manifolds of dimension $n \equiv 1 \pmod 4$

We would now like to use this type of invariant to generalize the notion of SK equivalence to an equivalence that respects SKK invariants. Now for, N_1, N_1', N_2, N_2', with $\partial N_1 = \partial N_2$ and $\partial N_1' = \partial N_2'$, an equivalence relation can be imposed on $\mathcal{M}_n$ (or $\mathcal{M}_n^O$)

Definition 2.7. **SKK-Equivalence** *can be defined by imposing all relations of the form* $(N_1 \cup_\psi -N_1') + (N_2 \cup_\varphi -N_2') = (N_1 \cup_\varphi -N_1') + (N_2 \cup_\psi -N_2')$ *on* $\mathcal{M}_n$.

Definition 2.8. *When we quotient* $\mathcal{M}_n$ *by the* **SKK equivalence** *given above, and take its Grothendieck completion, we obtain the group* $\mathbf{SKK_n}$*. When working with nonoriented manifolds, we obtain* $\mathbf{SKK_n^O}$*.*

These SKK groups and, to a somewhat lesser extent, the SK groups are closely connected to another more prominent equivalence relation. The notion of (co)bordism is prevalent in algebraic topology and is defined as follows:

Definition 2.9. *Two (oriented) n-manifolds,* M, M' *are* **cobordant** *if there exists an (oriented) (n+1)-dimensional manifold* Y *such that* $M + M' = \partial Y$*.*

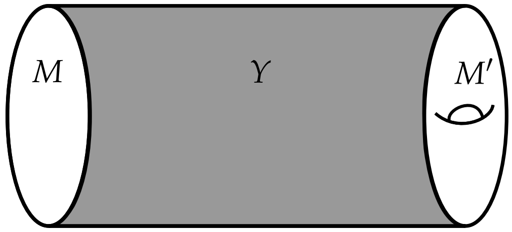

FIGURE 2. Cobordism Y of manifolds M and M'

Definition 2.10. *Let* $\mathbf{\Omega_n}$ *denote the quotient group after Grothendeick completion of cobordant oriented n-manifolds. Such a group is called the* **bordism group**. *In the non-oriented case, we denote the group* $\mathfrak{N}_\mathbf{n}$.

Now, that our relations have been defined, we would like to begin building and combining these notions. Primarily, we can ask what happens when both SK and bordism relations are imposed at the same time.

Definition 2.11. *Take* SK_n *and quotient by bordism relations to generate* $\overline{\mathbf{SK_n}}$. *This group can be formed equivalently by taking* Ω_n *and imposing the SK relations.* $\overline{\mathbf{SK}}^{\mathbf{O}}_{\mathbf{n}}$ *can be formed analogously using non-oriented relations.*

These relations clearly commute, and so we have the following diagram.

$$\begin{array}{ccc} \mathcal{M}_n & \longrightarrow & SK_n \\ \downarrow & & \downarrow \\ \Omega_n & \longrightarrow & \overline{SK}_n \end{array}$$

This construction allows us to study the differences between SK and bordism relations. By studying the kernels of these quotients, we can see how these relations differ. This gives rise to the following theorems.

Theorem 2.1. [KKNO73, Theorems 1.1, 1.2, 4.2] *Let* $I_n \subset SK_n$ *be the subgroup generated by the equivalence class* $[S^n]_{SK}$. *Let* $I'_n \subset SKK$ *be the subgroup generated by the equivalence class* $[S^n]_{SKK}$ *and* $F_n \subset \Omega_n$ *be the subgroup generated by all* $[M]_{\Omega_n}$ *where a representative M fibers over the circle. Equivalently this is the subgroup generated by the classes of mapping tori. Now the following three sequences are exact.*

$$0 \longrightarrow I_n \longrightarrow SK_n \longrightarrow \overline{SK}_n \longrightarrow 0 \tag{2.1}$$

$$0 \longrightarrow F_n \longrightarrow \Omega_n \longrightarrow \overline{SK}_n \longrightarrow 0 \tag{2.2}$$

$$0 \longrightarrow I'_n \longrightarrow SKK_n \longrightarrow \Omega_n \longrightarrow 0 \tag{2.3}$$

Additionally, these sequences are exact when each group is replaced with its non-oriented analog. I_n and I'_n have been computed as follows:

$$I_n = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.4}$$

for both oriented and non-oriented cases, and

$$I'_n = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}_2 & n \equiv 1 \ (\text{mod } 4) \\ 0 & n \equiv 3 \ (\text{mod } 4) \end{cases} \tag{2.5}$$

$$I'^{O}_n = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.6}$$

Additionally, 2.1 and 2.3 split in the oriented case.

We will only present the proofs of the exactness of the sequences, due to their geometrically enlightening nature. However, before presenting these proofs, I would like to make a note on the calculations of these sequences. These groups are computed by examining the values the invariants mentioned above take on specific manifolds. I_n is computed by studying Euler characteristic and signature, with I'_n proved similarly except for the $1 \pmod 4$ which uses the Kervaire Semicharacteristic. More in depth proofs are presented in [KKNO73]. To prove the exactness of these lemmas, we present the following lemmas from [KKNO73]:

Lemma 2.1. [KKNO73, Lemma 1.5 (i)] $[S^1]_{SK} = [0]_{SK}$

Proof. We will show that S^1 is SK-equivalent to $S^1 + S^1$. Let I_i denote an i^{th} copy of unit interval. In the context of the definition of SK-equivalence, we can consider $S^1 = (I_1 + I_3) \cup_\varphi (I_2 + I_4)$, where φ identifies endpoints by $\{1\}_i \sim \{0\}_{i+1}$. Now consider the identification

$$\psi : \begin{cases} \{1\}_1 \mapsto \{0\}_4 \\ \{0\}_1 \mapsto \{1\}_4 \\ \{1\}_2 \mapsto \{0\}_3 \\ \{0\}_2 \mapsto \{2\}_3 \end{cases}$$

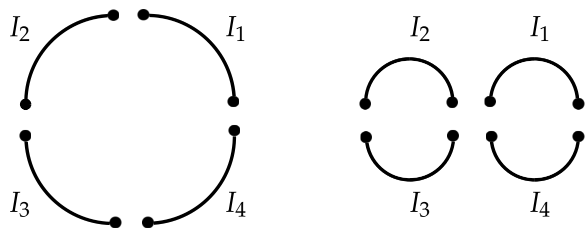

FIGURE 3. Demonstrating SK Equivalence of S^1 and $S^1 + S^1$

We see this funtion pairs the intervals off to form two circles. $S^1 + S^1 = (I_1 + I_3) \cup_\psi (I_2 + I_4)$ This is illustrated in Figure 3. Thus $[S^1]_{SK} = 2[S^1]_{SK}$ and so $[S^1]_{SK} = [0]_{SK}$. $\square$

Lemma 2.2. [KKNO73, Lemma 1.5 ii)] *If M fibers of S^n with fiber F, then $[M]_{SK} = [S^n]_{SK}[F]_{SK}$.*

Proof. Consider M which fibers over S^n with fiber F. Then recall $S^n = D^n \cup_{id} D^n$ Then $M = (F \times D^n) \cup_\varphi (F \times D^n)$, for some φ. However, this is then SK-equivalent to $(F \times D^n) \cup_{id} (F \times D^n) = F \times S^n$, so the M is SK-equivalent to $S^n \times F$ and so $[M]_{SK} = [S^n]_{SK}[F]_{SK}$ $\square$

Additionally, here we would like to introduce a special class of manifold called mapping tori. For a closed manifold F, consider the manifold $F \times [0,1]$. Note this manifold has boundary $F + F$. Now, in the same way S^1 can be viewed as the unit interval with endpoints identified, we can similarly identify the two boundary copies of F together using the identity to form $F \times S^1$. However, if we chose to use a different smooth diffeomorphism $\varphi : F \to F$ to glue along, the resulting manifold is called a **mapping torus** and is denoted T_φ. Specifically, we make the identification

$$T_\varphi = F \times [0,1]/((x,0) \sim (\varphi(x),1)).$$

For a simple example, consider the mobius band and the unit cylinder both having fiber $F = [0,1]$. The unit cylinder is the identity mapping torus, while the mobius band is the mapping torus with attaching map $\varphi(t) = (1-t)$. As described here, mapping tori clearly fiber over S^1, and thus by combining the above lemmas, we see that the mapping tori denote the zero class in SK.

Another construction which will be relevant is the **twisted double**. For two diffeomorphic manifolds N, N', with diffeomorphisms $f, g : N \to N'$, the twisted double $TD_{f,g}$, is defined as $(N \times [0,1] + N' \times [0,1]/((x,0) \sim (f(x),0), (x,1) \sim (g(x),1))$. A picture is shown in Figure 5. It is a known result that any twisted double is cobordant to the mapping torus $T(gf^{-1})$.

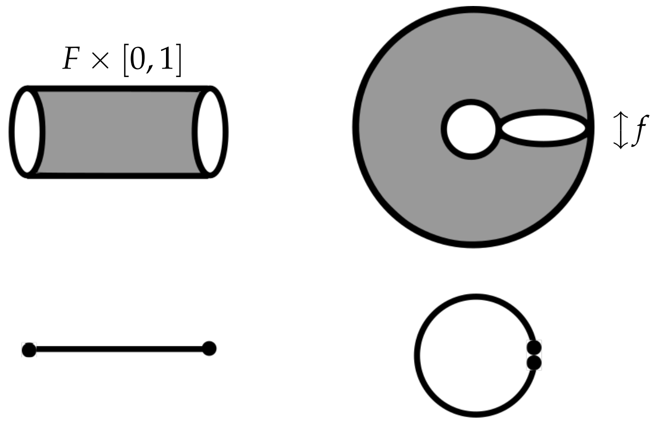

FIGURE 4. Construction of Mapping Torus

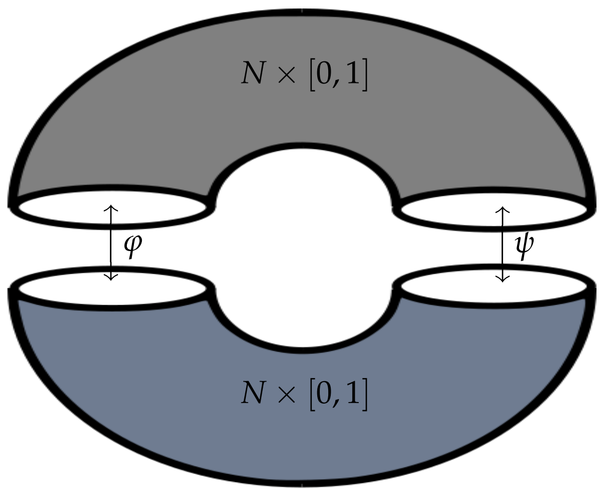

FIGURE 5. Twisted Double

Another notion that we will need is that of a geometric surgery. Surgery on a manifold M is a method of producing a new manifold M' in the following way:

Definition 2.12. *Let M be a manifold with an embedding of $S^k \times D^{n-k}$. Now take $cl.(M \setminus (S^k \times D^{n-k}))$ and paste in $D^{k+1} \times S^{n-k-1}$, via gluing along the boundary. Then the manifold $M' = cl.(M \setminus (S^k \times D^{n-k})) \cup (D^{k+1} \times S^{n-k-1})$ is produced from M by* **surgery of type $(\mathbf{k+1}, \mathbf{n-k})$.**

Note that $S^n = (S^k \times D^{n-k}) \cup (D^{k+1} \times S^{n-k-1})$, and thus both have the same boundary, and so this identification is consistent.

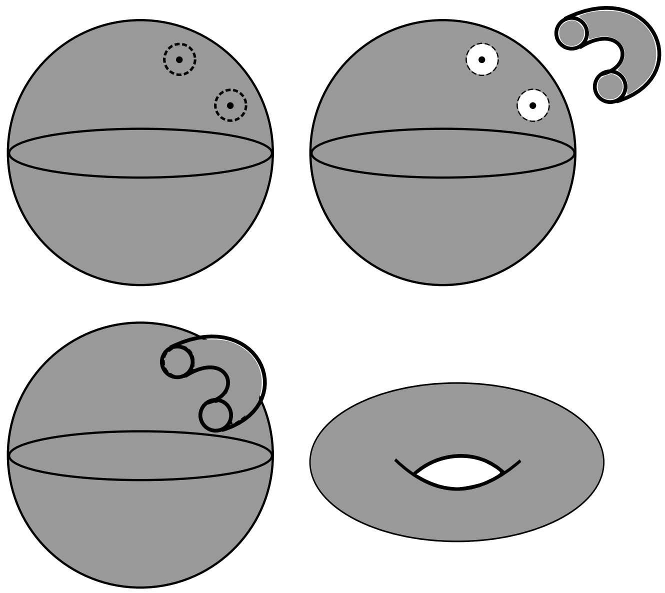

FIGURE 6. Surgery of Type $(1,2)$ on S^2

Example 2.2. *We present an example of surgery on S^2 of type $(0+1, 2-0)$. We take S^2 and look for an imbedding of $S^0 \times D^2$. This is two disjoint disks on S^2 with boundary $S^1 + S^1$. We now would like to glue the handle $S^1 \times D^1$, with is a hollow cylinder with boundary $S^1 + S^1$. By removing the two disks, and gluing in the handle, we produce a torus. A visualization of this example can be seen in Figure 2.2.*

Lemma 2.3. [KKNO73, Lemma 1.6] *If M' is a manifold obtained from surgery of type $(k+1, n-k)$ on a manifold M, then $[M]_{SK} + [S^n]_{SK} = [M']_{SK} + [S^k \times S^{n-k}]_{SK}$*

Proof. First recognize the following identities:

$$M + S^n = M \setminus (S^k \times D^{n-k}) \cup (S^k \times D^{n-k}) + (S^k \times D^{n-k}) \cup (D^{k+1} \times S^{n-k-1})$$

$$M' + S^k \times S^{n-k} = M \setminus (S^k \times D^{n-k}) \cup (D^{k+1} \times S^{n-k-1}) + (S^k \times D^{n-k}) \cup (S^k \times D^{n-k})$$

Then, we view manifolds N as the disjoint union of the first components and N' as the disjoint union of the second:

$$N = M \setminus (S^k \times D^{n-k}) + (S^k \times D^{n-k})$$

$$N' = (S^k \times D^{n-k}) + (D^{k+1} \times S^{n-k-1})$$

Then $M + S^n = N \cup_{id} N'$ and $M' + S^k \times S^{n-k} = N \cup_t N'$, where t swaps the components being glued, and so we now see these are SK-equivalent. $\square$

Now, by setting $M = S^n$. We see $2[S^n]_{SK} = [S^{k+1} \times S^{n-k-1}]_{SK} + [S^k \times S^{n-k}]_{SK}$ and now set $k = 0$ and by induction, we see the following

Corollary 2.1. [KKNO73, Cor. 1.7]

$$[S^k \times S^{n-k}]_{SK} = \begin{cases} 2[S^n]_{SK} & = k \text{ even} \\ 0 & = k \text{ odd} \end{cases}$$

Corollary 2.2. [KKNO73, Cor. 1.8] *When M and M' are related by surgery,*

$$[M]_{SK} = [M']_{SK} + (-1)^{k+1}[S^n]_{SK}$$

We now have the sufficient tools to prove the exactness of sequences 2.1, 2.2, and 2.3 from Theorem 2.1.

Proof of (2.1). First, note that the our two maps of interest are clearly injective (by inclusion) and surjective (by quotient) respectively. Thus it suffices to show that $I_n = \operatorname{Ker}(SK_n \to \overline{SK}_n)$. The first inclusion, $I_n \subset \operatorname{Ker}(SK_n \to \overline{SK}_n)$, is easy to see as all S^n are cobordant to the empty set as $\partial D^{n+1} = S^n + \varnothing$. Then it suffices to show $I_n \supset \operatorname{Ker}(SK_n \to \overline{SK}_n)$. To show this, we take two manifolds related via bordism, and compare their difference in SK_n. It is a fact that bordisms can be decomposed into a series of surgeries. That is to say, for cobordant M, M', there is a finite sequence of surgery operations and manifolds M_i such that M' can be can be obtained from M by interated surgeries on M_i. Then when looking at $[M]_{SK} - [M']_{SK}$, the difference at each stage of surgery is $\pm[S^n]$ by Cor. 2.2, so therefore $[M]_{SK} - [M']_{SK} = n[S^n]_{SK}$ for some constant n, determined by the sequence of surgeries. So the difference of any two cobordant manifolds in SK_n is some sum of spheres, $\operatorname{Ker}(SK_n \to \overline{SK}_n) \subset I_n$. □

Proof of (2.2). First note that an equivalent formulation of F_n is as the collection of mapping tori cobordism classes, as fibering over S^1 is equivalent to being a mapping torus as described above for some φ. Now by construction F^n includes into Ω_n. Additionally, by construction, Ω_n surjects onto $\overline{SK}$. Additionally, $F^n \subset Ker(\Omega_n \to \overline{SK})$, as the SK class of all mapping tori is 0.

Now we must show the reverse inclusion. This can be accomplished by showing the difference as cobordism classes of two SK-equivalent manifolds is a mapping tori. Let $M = N \cup_\varphi N'$ and $M' = N \cup_\psi N'$ be two SK-equivalent manifolds. Now let us construct a bordism by taking $N \times [0,1]$ and $N' \times [0,1]$ and gluing by φ on the first third, ψ on the last third, and leaving the middle unglued.

The boundary of this manifold will be $M + M' + (\partial N \times [1/3, 2/3]) \cup_{\varphi,\psi} (\partial N' \times [1/3, 2/3])$. This last manifold is equivalent to a twisted double as $\partial N'$ is diffeomorphic to ∂N, so as mentioned before this will be cobordant to $T(\psi\varphi^{-1})$. with

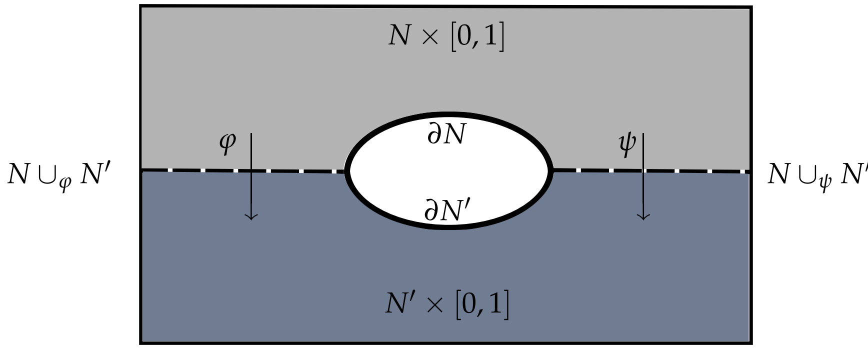

FIGURE 7. $[M]_\Omega - [M']_\Omega = [T(\varphi\psi^{-1})]_\Omega$

fiber ∂N. Therefore if M, M' are SK-equivalent manifolds, then $[M]_\Omega - [M']_\Omega = [T(\psi\varphi^{-1})]_\Omega$. □

Proof of (2.3). First, we must show there is a well-defined surjection from SKK_n to the respective bordism group. To do this, we will note that bordism is an SKK-invariant. Note that as a consequence of the last proof, $[M]_\Omega - [M']_\Omega = [T(\varphi\psi^{-1})]_\Omega$. The class of the mapping torus is entirely dependent on the boundary maps, and thus bordism defines an SKK-invariant. Therefore, assigning bordism class is a well-defined mapping. Further, I'_n includes into Ω_n, and $I'_n \subset \mathrm{Ker}(SKK_n \to \Omega_n)$, for similar reasons as in (2.1). Therefore, it suffices to show that the rest of the proof of Lemma 2.3 requires only SKK-equivalence, and then all following corollaries will hold, and this proof will proceed as in the proof of (2.1). This is easy to see by introducing an additional term. Notice that as defined in the proof of Lemma 2.3 $\partial N = \partial N' = 2\partial(S^k \times D^{n-k})$. Now let $Q = S^k \times D^{n-k}$. Notice that $2Q$ has the same boundary as N and N'. Now notice that $2Q \cup_{id} 2Q = 2Q \cup_t 2Q$, as each is just the disjoint union of 2 copies of $\mathcal{D}Q$, the double of Q. Then via the relation

$$[N \cup_t N']_{SKK} + [Q \cup_{id} 2Q]_{SKK} = [2Q \cup_t 2Q]_{SKK} + [N \cup_{id} N']_{SKK}$$

and that $2Q \cup_{id} 2Q = 2Q \cup_t 2Q$, we get $[N \cup_t N']_{SKK} = [N \cup_{id} N']_{SKK}$, and thus all other results shall follow. □

The calculations of the various kernels is given in [KKNO73] and will not be replicated here.

An important note for the above proofs is that at no step was the argument itself dependent on the orientablity or non-orientability of each manifold. Therefore, so long as the mappings and gluings at each step are formulated in such a way that they respect these conditions, the results will hold for both cases.

2.1. **SKV and Vector Field Cobordism.** In this section, we discuss some notions related to SKK-groups. The first provides an alternative interpretation of SKK in terms of bordism. SKK can also be interpreted as **Vector Field Cobordism**, where the bordism relation is modified to require the the bordism admit a non-zero vector field inward normal on M and outward normal on M'. In this context, we can give an alternate view of Sequence 2.3 and the splitting induced by the Kervaire semicharacteristic.

$$\begin{tikzcd} 0 \arrow[r] & I'_n \arrow[r] & SKK_n \arrow[l, bend right, dashed] \arrow[r] & \Omega_n \arrow[r] & 0 \end{tikzcd}$$

As presented above, the Kervaire semicharacteristic can be interpreted as the sole factor differentiating classical bordism from vectorfield bordism. As such, Prevot in [Pre80] introduces a modification of SKK to remove the 2-torsion induce by the Kervaire Semicharacteristic. He defines the below relation of an SKV (or SK-Vier) operation.

Definition 2.13. *An* **SKV-operation** *on an n-manifold* M_1 *produces a new n-manifold* M_2 *if the following hold:*

- *There exists an n-manifold P and four disjoint imbeddings ι_i of a closed (n-1)-manifold T, such that $\partial P = \amalg_{i=0}^{4} \iota_i(T)$.*
- *There are diffeomorphisms Φ, Ψ such that*

$$\Phi : M \to P/{\sim_1}$$

$$\Psi : N \to P/{\sim_2}$$

where $\sim_1$ makes identifications $\iota_1(t) \sim \iota_2(t)$ and $\iota_3(t) \sim \iota_4(t)$ and $\sim_2$ makes identifications $\iota_1(t) \sim \iota_4(t)$ and $\iota_2(t) \sim \iota_3(t)$.

Definition 2.14. *Two manifolds are* **SKV-equivalent** *if there are a finite number of SKV steps which produce one from the other.*

Definition 2.15. $\mathbf{SKV_n}$ *is the group obtained by quotienting $\mathcal{M}_n$ by the SKV equivalence and taking the Grothendieck completion. Similarly, we may obtain* $\mathbf{SKV_n^O}$.

We now present results without proof from [Pre80] which will be relevant to modify Theorem 2.9

Proposition 2.1 ([Pre80])**.** *There is an isomorphism*

$$SKK_n \to \begin{cases} SKV_n & n \not\equiv 1 \pmod 4 \\ SKV_n \oplus \mathbb{Z}_2 & n \equiv 1 \pmod 4 \end{cases} \tag{2.7}$$

Note that in the non-oriented case, the Kervaire semicharacteristic is just the trivial map, and so induces no torsion, and we have $SKV_n^O \cong SKK_n^O$.

2.2. **Results.** We would now like to present some non-equivariant results. We would like to study the map $SKK \to SK$, to understand the relation between these two groups. However, as we see above this will be heavily case dependent, as the kernel of the two sequences 2.1 and 2.3 change depending on the congruence class of $n \pmod 4$. We now present results that hold specifically for all but $n \equiv 1 \pmod 4$.

Theorem 2.3. *We present the new sequences:*

$$0 \longrightarrow F_n \longrightarrow SKK_n \longrightarrow SK_n \longrightarrow 0 \tag{2.8}$$

This sequence is exact for $n \not\equiv 1 \pmod 4$ in the oriented case, and all n in the non-oriented case. This is a consequence of the below braid commuting, when $n \not\equiv 1 \pmod 4$.

$$\begin{tikzcd}
 & I_n \cong I'_n \arrow[dr] \arrow[rr, bend left] & & SK_n \arrow[dr] \arrow[rr, bend left] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & SKK_n \arrow[ur] \arrow[dr] & & \overline{SK}_n \arrow[ur] \arrow[dr] & & 0 \\
 & F_n \arrow[ur] \arrow[rr, bend right] & & \Omega_n \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] &
\end{tikzcd} \tag{2.9}$$

Proof. To check commutativity of the diagram, we must confirm the squares around SKK_n commute, as the other squares trivially commute. Additionally, the first square commutes also trivially. Now, let us check the top and bottom triangles. Notice that $[S^n]_{SKK} = [S^n]_{SK}$, so by including $[S^n]_{SKK}$ into SKK_n and then identifying by SK_n relations, this is equivalent to just sending $[S^n]_{SKK}$ to $[S^n]_{SK}$ in SK_n. A similar argument holds for the lower triangle, where we see that the collection of mapping tori F_n can be regarded as either their SKK_n class or as their cobordism class. For the middle diamond, notice that this is a sequence of quotients, which hold regardless of order. That is taking an SKK class and sending it to its SK class and then its bordism class is equivalent to sending an SKK class to its bordism class and then its SK class. Thus the above diagram commutes. To apply the braid lemma to the diagram above, we must now show that 2.3 is a chain complex. By noting that F_n was all manifolds which fiber of S^1 and take the zero class in SK, we see $F_n \subset \mathrm{Ker}(SKK_n \to SK_n)$, which shows 2.3 is a chain complex and we may now apply the braid lemma, which completes the proof. $\square$

We see that the sole obstrution to the above braid commuting for all n is the $\mathbb{Z}_2$ torsion caused by the Kervaire semicharacteristic. As we saw above, the notion of SKV allows us to discard the torsion, which leads to the following theorem:

Theorem 2.4. *By using the notion of SKV in place of SKK, we have the sequence for all n, in both the oriented an non-oriented cases:*

$$0 \longrightarrow F_n \longrightarrow SKV_n \longrightarrow SK_n \longrightarrow 0 \tag{2.10}$$

This is a consequence of the braid below commuting for all n. This follows from the proof of Theorem 2.3.

$$\begin{tikzcd}
 & I_n \arrow[dr] \arrow[rr, bend left] & & SK_n \arrow[dr] \arrow[rr, bend left] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & SKV_n \arrow[ur] \arrow[dr] & & \overline{SK}_n \arrow[ur] \arrow[dr] & & 0 \\
 & F_n \arrow[ur] \arrow[rr, bend right] & & \Omega_n \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] &
\end{tikzcd} \tag{2.11}$$

3. G-Spaces and Equivariant Mappings

Later in this paper, we would like to examine *equivariant SK*, *SKK*, and bordism relations. To do this, we must introduce the need background to discuss equivariance and its effects on the relations. We will be looking at a special class of manifolds, which will be called G-manifolds, which come equipped with a group action. This action permutes points within the space in some continuous way. Using this construction, we have a more dynamic and interesting object of study. The maps that respect such an action are said to be equivariant. Precise definitions are given below.

Definition 3.1. *A* **G-action** *given by a group G acting on a topological space X is a map $\Theta : G \times X \to X$ satisfies the following conditions:*

- $\Theta(h, \Theta(g, x)) = \Theta(hg, x)$
- $\Theta(e, x) = x$

A space equipped with a G-action is called a G-space.

Equivalently, an action can be formulated as a map $\theta : G \to Homeo(X)$, where $\theta(g) = \Theta(g, -)$. We will denote $\Theta(g, x) = gx$. If θ is injective, we say that it is an **effective** *action.*

Definition 3.2. *A* **a G-map** *is a map φ between two spaces, X and Y, each with a G-action. Such a map is* **equivariant** *if the following diagram commutes:*

$$\begin{array}{ccc} G \times X & \xrightarrow{id \times \varphi} & G \times Y \\ \Big\downarrow{\scriptstyle \Theta_X} & & \Big\downarrow{\scriptstyle \Theta_Y} \\ X & \xrightarrow{\quad \varphi \quad} & Y \end{array}$$

Now, we would like to define some of the specific constructions and objects used in the study of equivariant manifolds and equivariant study in general.

Definition 3.3. *Let $x \in X$. The collection of points x is sent to under the action of G is called the* **orbit** *of x (denoted* **G(x)** *). That is $G(x) = \{gx \mid g \in G\}$*

Definition 3.4. *A set $S \subset X$ is* **H-invariant** *for a subgroup $H \subset G$ if $H(S) = S$, where $H(S) = \{hs | h \in H, s \in S\}$. This is equivalent to being the union of the orbits of all elements in S.*

Definition 3.5. *The X/G is the quotient space of X created by identifying $x \sim gx$ for all g in G. Elements $\overline{x} \in X/G$ represent $G(x)$, hence the terminology. The quotient map $\pi : X \to X/G$ is natural.*

Definition 3.6. *The* **isotropy subgroup** *(denoted $\mathbf{G_x}$) of G for a point x is the collection of elements in G which fix a given x. That is $G_x = \{g \in G \mid gx = x\}$*

Definition 3.7. *A* **fixed point** *of X under a G-action is a point x such that $gx=x$ for all $g \in G$. The collection of all fixed points is denoted X^G*

The previous notion specifically is very important. Fixed points carry a large amount of information relating to the G-action and its connection to the underlying manifold. Additionally, fixed points sets are important in extending topological invariants such as Euler characteristic, signature, and Kervaire semicharacteristic.

The above definitions allow us to study the group action in a somewhat global sense, that is how it acts on the entire manifold. We would like to introduce tools that allow us to study the local action of the group. The first of these is the twisted product, defined below:

Definition 3.8. *We will now define the* ***twisted product***, $\mathbf{G \times_H A}$. *Let H be a compact subgroup of G and let A be some space which has an H-action. H acts on $G \times A$ by $(h, (g, a)) \mapsto (gh^{-1}, ha)$. We will denote the orbit space as $G \times_H A$. For a point (g, a) the orbit is $[g, a] = \{(gh^{-1}, ha) | h \in H\}$. As classes $[g, a] = [g', a']$ if $\exists\, h \in H$, such that $g' = gh^{-1}$ and $a' = ha$. The map $a \mapsto [e, a]$ defines an embedding $A \hookrightarrow G \times_H A$. Additionally, $G \times_H A$ inherits a left G-action, by $(g, [g', a]) \mapsto [gg', a]$*

This definition above helps us give the definition of a slice type. The slice type at a point describes the local action of G, and how it interacts with the local Euclidean structure. The slice type conveys the information that will be important in defining equivariant SK-relations.

The definition below is provided by Soren Illman:

Definition 3.9. *Let H be a subgroup of G and H-invariant $S \subset X$, a smooth submanifold, with X being a smooth G-manifold. S is a* **smooth H-slice** *if G(S) is open in X, and*

$$\mu : G \times_H S \to G(S), [g, s] \mapsto gs$$

is a smooth diffeomorphism.

Lemma 3.1. *Let S be a smooth submanifold of X. Then the following are equivalent:*

(1) *S is a smooth H-slice in X.*
(2) *$G(S)$ is open in X and there exists a G-equivariant smooth map $r : G(S) \to G/H$ such that $r^{-1}(eH) = S$.*

Theorem 3.1 (The Slice Theorem)**.** *Suppose a compact Lie group G acts smoothly on a manifold X. Let $x \in X$. Then there is vector space V_x on which the isotropy group G_x acts linearly and a G-embedding $G \times_{G_x} V_x \to X$ onto an open set which sends $[g, 0]$ to gx.*

There are several things to note in this theorem:

- We have already noted in Lemma 2 that the map $G/G_x \to X; g \mapsto gx$ is a smooth embedding.
- Recall that $G \times_{G_x} pt \simeq G/G_x$. So the map $G/G_x \to X$ extends to an invariant neighbourhood of G/G_x in $G \times_{G_x} V_x$ viewed as a zero section.
- The slice theorem implies that there is an equivariant diffeomorphism from $G \times_{G_x} V_x$ to its image in X.
- Note that this image in X contains the orbit of the point x, $G(x)$.
- One way to describe what a "slice at x" is, is as the image of $\{e\} \times V_x$ in X. Exercise: Why does this description match the definition of slice given above?

Now, where does this vector space V_x come from? To understand this, we must first understand the idea of a representation. A G-action on a vectorspace V can be "represented" by a group homomorphism $\sigma : G \to GL(V)$, where an element g determines a matrix transformation on V. Such an action is called a representation of V. A subrepresentation of V is a subvectorspace which is also closed under the G-action. All representations can be decomposed into a direct sum of irreducible subrepresentations (unique up to isomorphism). In this way, any representation can be broken up into two parts: the subrepresentations with trivial action and the subrepresentations with non-trivial action. [Smi]

Now by [Bre72], we have another phrasing of the slice theorem, which states each point in X has a G_x-neighborhood U_x, which is topologically homeomorphic to $\mathbb{R}^n$. This then gives a G_x representation on U_x, which as we discussed above, can be decomposed into $\mathbb{R}^p \oplus V_x$, where $\{0\}$ is the only fixed point of V_x. The subrepresentation V_x carries with it all of the information about the G_x action. Then the twisted product $G \times_{G_x} V_x$ then gives all information relating to the G-action locally. This notion intuitively defines the **slice type** at a point x. The slice type is the conjugacy class of G_x and its representation $\sigma : G_x \to V_x$. The slice types entirely determine the local structure of a G-manifold, as by the slice theorem above, each point has some neighborhood described in this way.

We now have sufficient material to begin discussing equivariant analogs of the topics discussed in Section 2.

4. Equivariant SK

The idea of SK-equivalence and bordism can be extended to the equivariant case. The notions generalize as expected, with a few modifications. To start with, we will disregard orientation and focus on the non-oriented case. The notion of an oriented G-manifold complicates matters significantly, as the orientation is (locally) determined by a small neighborhood around a given point. The G-action must then respect the orientation locally, and this would be reflected in the twisted product and the slice type. Oriented results will be presented in Section 5.

First consider the collection of n-dimensional G-Manifolds $\mathcal{M}_n^O[G]$. To produce equivariant SK groups, denoted $SK_n^O[G]$, factor all G-manifolds the cut-paste relations described above, with the added condition that the diffeomorphisms are now G-equivariant. The $SKK_n^O[G]$ groups are defined with the same added condition. To generate equivariant bordism groups, which we will now denote by $\mathfrak{N}_n[G]$, factor $\mathcal{M}_n^O[G]$ by the G-cobordism relation, which says that two manifolds are G-cobordant if they are the boundary of an $(n+1)$-dimensional compact G-manifold. We additionally ask that the manifolds and bordisms have no isolated fixed points, which we will call a *nice* bordism after [WW88], and will be denoted with $\Omega_n^\bullet$ or $\mathfrak{N}_n^\bullet$. One can then ask if all of the above exact sequences carry through to the equivariant case, and if they do, can a splitting be constructed to determine all such equivariant SK and SKK invariants, as was the case for the non-equivariant sequences. We then have the following theorem.

Theorem 4.1. *For G of odd finite order, the below sequences are exact, with surjections defined by factoring with equivariant SK and nonoriented nice cobordism relations:*

$$0 \longrightarrow I_n^O[G] \longrightarrow SK_n^O[G] \longrightarrow \overline{SK}_n^O[G] \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow F_n^O[G] \longrightarrow \mathfrak{N}_n^\bullet[G] \longrightarrow \overline{SK}_n^O[G] \longrightarrow 0 \tag{4.2}$$

$$0 \longrightarrow I'^O_n[G] \longrightarrow SKK_n^O[G] \longrightarrow \mathfrak{N}_n^\bullet[G] \longrightarrow 0 \tag{4.3}$$

where, $I_n^O[G], F_n^O[G]$, and $I'^O_n[G]$ are equivariant analogs of I_n^O, F_n^O, and I'^O_n and are defined later.

4.1 and 4.2 are (implicitly) proved in [Har04], and 4.3 is proved in [WW88]. These will be proven by adapting the non-equivariant proofs to the equivariant case. To successfully generalize the non-equivariant case to prove 4.1 and 4.3, we need to develop the idea of G-surgery.

The notion of surgery in the classical sense involves removing part of the manifold diffeomorphic to $S^k \times D^{n-k}$ and gluing along boundary the complimentary $D^{k+1} \times S^{n-k-1}$. We must now take this idea and convert it to an equivariant setting, such that the surgery operation respects the G-action on the manifold. To do this we must define suitable notions of the disk and the sphere for G-manifold. Let V be a vector space, and let $G \to GL(V)$ be a representation of V.

Definition 4.1. *The* **unit disk in the representation** *$D(V)$ is a G-invariant unit disk in V, $D(V) = \{v \in V : ||v||) \leq 1\}$. The* **unit sphere in the representation** *$S(V)$ is a G-invariant unit sphere in V, $S(V) = \{v \in V : ||v||) = 1\}$. Note that the dimension of $S(V)$ will be dim(V)-1.*

In this way, a G-surgery can be defined. Given an n-dimensional G-manifold, let $\varphi : G \times_H (D(V \oplus \mathbb{R}^p) \times S(\mathbb{R}^q)) \to M$, be a smooth G-embedding onto a G-invariant regular submanifold of M, with $p + q = n - \dim(V) + 1$ Then the manifold M' obtained from G-surgery on M is defined in the following way:

$$M' = M \setminus \varphi(G \times_H (D(V \oplus \mathbb{R}^p) \times S(\mathbb{R}^q))) \cup G \times_H (S(V \oplus \mathbb{R}^p) \times D(\mathbb{R}^q))$$

Now that G-surgery has been defined, we may now sketch the proofs of 4.1 and 4.3.

Proof of (4.1) and (4.3). A nice G-bordism L is obtained from G-surgery by taking $M \times [0,1]$ and performing the surgery on $M \times \{1\}$. For finite G, G-bordisms are all given by a sequence of G-surgeries, which is analogous to the non-equivariant case. Then Lemma 2.3 still holds equivariantly, and we see that the difference of any two $SK_n[G]$ (or $SKK_n[G]$) equivalent manifolds is $\sum(-1)^i[G \times_H S(V_i \oplus \mathbb{R}^{p+q})]$, where each summand comes from a single surgery step. Then the kernel $I_n^O[G]$ is the collection in $SK_n^O[G]$ of all $[G \times_H S(V_i \oplus \mathbb{R}^{p+q})]_{SK[G]}$ and $I'^O_n[G]$ is the collection in $SKK_n^O[G]$ of $[G \times_H S(V_i \oplus \mathbb{R}^{p+q})]_{SKK[G]}$.

In this context, we can say that I_n^O and I'^O_n can be defined:

$$I_n[G] = \{[G \times_H S(V_i \oplus \mathbb{R}^{p+q})]_{SK[G]} | \text{All Representations } V_i\}$$

$$I'_n[G] = \{[G \times_H S(V_i \oplus \mathbb{R}^{p+q})]_{SKK[G]} | \text{All Representations } V_i\}$$

where $p + q + \dim(V) = n + 1$ □

This proof is only explictly given in [Kom87]. In [Har04], the kernel of the surjection is computed, and no exact sequence is given. Here, $\text{Ker}(SK_n^O[G] \to \overline{SK}_n^O[G]) = I_n^O[G] = 2SK_n[G]$ for even dimensions and 0 for odd. It is to be noted that this alternative formulation of the kernel only holds when the non-oriented cobordism relation is imposed, as here, each class is its own inverse via the trivial bordism, and so this suffices as the only relation.

For 4.2, [Har04] once again does not explicitly give the exact sequence. However, he does compute and characterize $\text{Ker}(\mathfrak{N}_n[G] \to \overline{SK}_n^O[G])$ as a subgroup of $\mathfrak{N}_n[G]$. Once again, we may examine the nonequivariant proof to find an illuminating route for the equivariant case.

Proof of (4.2). Using the construction in the non-equivariant proof, we see that there is an analog to a mapping tori in the equivariant setting, that is G-manifolds which fiber over S^1 whose action permutes within fibers. We can see this as under the bordism construction, the G-action acted trivially on the interval, and so for the $[1/3, 2/3]$ subinterval, the G-action on $T(\varphi\psi^{-1})$ will act only on the fibers ∂N. Thus the difference of any two $SK_n[G]$ classes of manifolds up to cobordism will be these equivariant mapping tori. That is to say, $[M]_{\mathfrak{N}[G]} - [N]_{\mathfrak{N}[G]} = [T(\varphi\psi^{-1})_G]_{\mathfrak{N}[G]}$. This gives $\text{Ker}(\mathfrak{N}_n[G] \to \overline{SK}_n[G]) \subset F_n[G]$. Additionally, all such fiberings over S^1 take the zero class in $\overline{SK}_n[G]$ under equivariant cut paste relations by the same argument in as in Lemma 2.1. Then we see all such fiberings entirely describe the kernel of the surjection. □

We can also define an equivariant notion of Euler characteristic and Kervaire semicharacteristic. Let M be an n-dimensional G-manifold, and let L be some subgroup of G with representation σ. Then we define the equivariant Euler characteristic of M, $\chi_\sigma^L(M)$, to be the classical Euler characteristic on the fixed point set of M under the action L with representation σ. That is to say $\chi_\sigma^L(M) = \chi(M_\sigma^L)$. The same holds for Kervaire semicharacteristic k_σ^L. Additionally, as this is defined in relation to the classical construction, the classical properties hold, such as the additive structure, and that $\chi_\sigma^G(M) = 0$ for closed odd dimensional M.

4.1. **Notes on the Non-Oriented Case.** The non-oriented case introduces several interesting results. First, we have a result from [Kom05], that if $[M]_{SK} = [2x]_{SK}$, then M is cobordant to some G-equivariant fibering over S^1. We can see this quite easily now that the proper kernels have been identified. Clearly $[2x]_{SK[G]} \mapsto [0]_{\overline{SK}[G]}$, as under the non-oriented cobordism relation each element is its own inverse. As the map is well defined $[M]_{SK[G]} \mapsto [0]_{\overline{SK}[G]}$. However, as the square

$$\begin{array}{ccc} \mathcal{M}_n^O[G] & \longrightarrow & SK_n^O[G] \\ \downarrow & & \downarrow \\ \mathfrak{N}_n[G] & \longrightarrow & \overline{SK}_n^O[G] \end{array}$$

commutes, $[M]_{\mathfrak{N}[G]} \mapsto [0]_{\overline{SK}[G]}$, and thus $M \in \mathrm{Ker}(\mathfrak{N}_[G] \to \overline{SK}_n[G]) = F_n[G]$, and thus is cobordant to a G-fibering over S^1. This proves the result in [Kom05].

We would also like to perform a sanity check as we saw that $[M]_{SK[G]} \mapsto [0]_{\overline{SK}_n[G]}$, but $[M]_{SK[G]}$ was not explicitly a sum of spheres of representations . Thus we would like to confirm $[M]_{SK[G]} = \sum [G \times_H S(V)]_{SK[G]}$ as $[M]_{SK} \in I_n[G]$. This can also be easily seen by examining the result in [KKNO73, pg. 57] which gives the explicit inverse of an (oriented) SK-class $[N]_{SK}$.

$$[N]_{SK}^{-1} = [-N]_{SK} + \chi(M)[S^n]_{SK}$$

This result also carries over under equivariance. Here $[x]_{SK[G]}^{-1} = [-x]_{SK[G]} + \sum [G \times_H S(V)]_{SK[G]}$. Then (in $SK[G]$)

$$0 = [x] + [x]^{-1} = [x] + [-x] + \sum [G \times_H S(V)]$$

However, when forget orientation(as recall this is the non-oriented case), $[x] = [-x]$, and thus

$$\begin{aligned} 0 &= [2x] + \sum [G \times_H S(V)] \\ [2x] &= -\sum [G \times_H S(V)] \end{aligned}$$

and we see that $[M]_{SK[G]}$ was some sum of unit spheres of some set of representations.

4.2. **Equivariant SKV.** Such a generalization is defined in [Pre80] in the obvious way. The SKV relation above is now defined using G-manifolds and G-equivariant gluing maps. Additionally, we also define an F-free G-manifold. Let F be a family of subgroups of G such that if $K \subset H$ and $H \in F$, then $K \in F$. A G-manifold is F-free if all G_x are conjugate to some member of F. The SKV groups are be denoted by $SKV_n[G,F]$, and similarly for bordism. We also ask that we have the notion of a nice cobordism defined above. The results in [Pre80] present an analog to Equation 2.7 for the equivariant case, namely:

$$SKK_n[G,F] = \begin{cases} SKV_n[G,F] & n \text{ even} \\ SKV_n[G,F] \oplus E_n[G,F] & n \text{ odd} \end{cases} \tag{4.4}$$

where $E_n[G,F]$ generalizes the $\mathbb{Z}_2$ torsion term in equation 2.7. To define $E_n[G,F]$, we must first study why exactly the $\mathbb{Z}_2$ term appears in the first place. To study this, we must examine the Kervaire semicharacteristic amd other invariants. As described above, non-equivariant SK class is entirely determined by Euler characteristic and signature invariant. SKK class is determined entirely by bordism and Euler characteristic in all cases except $n \equiv 1 \pmod 4$. In this case, the Kervaire semicharacteristic determines another invariant that induces a splitting on SKK. If $E_n[G,F]$ is meant to generalize this torsion, then it must in some way relate to the equivariant notions of Euler characteristic and Kervaire semicharacteristic. To do this, we define an equivalence relation on $\mathcal{M}_n[G]$, which states that $M \sim M'$ if $\chi_\sigma^G(M) = \chi_\sigma^G(M')$and $k_\sigma^G(M) = k_\sigma^G(M')$. The group obtained after the Grothendieck construction is then $E_n[G]$.

5. Results

First, we present exact sequences for the oriented, equivariant case. This can be done by asking that our G-actions are globally orientation preserving, and that our gluing maps are orientation preserving, equivariant diffeomorphisms.

Theorem 5.1. *For G of odd finite order, the below sequences are exact, with surjections defined by factoring with equivariant SK and oriented nice cobordism relations:*

$$0 \longrightarrow I_n[G] \longrightarrow SK_n[G] \longrightarrow \overline{SK}_n[G] \longrightarrow 0 \tag{5.1}$$

$$0 \longrightarrow F_n[G] \longrightarrow \Omega_n^\bullet[G] \longrightarrow \overline{SK}_n[G] \longrightarrow 0 \tag{5.2}$$

$$0 \longrightarrow I'_n[G] \longrightarrow SKK_n[G] \longrightarrow \Omega_n^\bullet[G] \longrightarrow 0 \tag{5.3}$$

where, $I_n[G], F_n[G]$, and $I'_n[G]$ are equivariant analogs of I_n, F_n, and I'_n.

Proof. The proofs of these follow using the same arguments for the non-oriented cases, with the added condition that the G-action and gluing diffeormorphisms both preserve orientation. We note that the proving the exactness of each sequence above does not depend on either the orientatedness or non-orientedness of the manifolds we are working with. □

Now, we have also constructed an equivariant analog of the sequence from Theorem 2.3, for both the oriented and non-oriented case. But first we must present several lemmas.

Lemma 5.1. *If M and M' are $SK[G]$-equivalent $2k$-dimensional G-manifolds, then $\chi_\sigma^G(M) = \chi_\sigma^G(M') = \chi_\sigma^G(Y)$, where Y is the bordism constructed in the proof of 4.2.*

Proof. The first equality holds immediately, as χ_σ^G is an $SK[G]$ invariant. Then let us consider the bordism Y as pictured in Figure 5. We partition this space into two spaces A and B, where A is the portion of the bordism along $[0, 1/2+\epsilon]$. B is the complimentary portion of the bordism along $[1/2-\epsilon, 1]$. Using the additivity of χ_σ^G, we see $\chi_\sigma^G(Y) = \chi_\sigma^G(A) + \chi_\sigma^G(B) - \chi_\sigma^G(A \cap B)$. We see that $\chi_\sigma^G(A) = \chi_\sigma^G(M)$ via retraction along the interval and compatibility with the equivariant gluing, and likewise for $\chi_\sigma^G(B) = \chi_\sigma^G(M')$. We can see that $A \cap B = N + N'$, so $\chi_\sigma^G(A \cap B) = \chi_\sigma^G(N) + \chi_\sigma^G(N')$. Then by replacement we see that $\chi_\sigma^G(Y) = \chi_\sigma^G(M) + \chi_\sigma^G(M') - (\chi_\sigma^G(N) + \chi_\sigma^G(N'))$. Now, let us examine $\chi_\sigma^G(M)$. Partition M into N_U and N'_U, where these are N and N' each with an added collared neighborhood around the gluing at ∂N. Then we see that these allow us to simplify as follows: $\chi_\sigma^G(M) = \chi_\sigma^G(N) + \chi_\sigma^G(N') - \chi_\sigma^G(\partial N)$. Then by substitution and cancellation again we find that $\chi_\sigma^G(Y) = \chi_\sigma^G(M') + \chi_\sigma^G(\partial N)$. Now ∂N is a closed, odd-dimensional, G-manifold, and thus $\chi_\sigma^G(\partial N) = 0$, so we have that $\chi_\sigma^G(Y) = \chi_\sigma^G(M')$ and so we are done. □

A second proof is also presented:

Proof. For this proof, once again let Y be the bordism as described in Figure 9. We construct the double $2Y$ by gluing via identity along boundary of two copies of Y. This is a closed $n+1$-dimensional manifold. Then, by using collared neighborhoods around each copy of Y, we get $\chi_\sigma^G(2Y) = 2\chi_\sigma^G(Y) - \chi_\sigma^G(\partial Y)$. As $\partial Y = M + M' + T(\varphi\psi^{-1})_G$, we get $\chi_\sigma^G(2Y) = 2\chi_\sigma^G(Y) - (\chi_\sigma^G(M) + \chi_\sigma^G(M') + \chi_\sigma^G(T(\varphi\psi^{-1})_G)))$. As $[T(\varphi\psi^{-1})_G))]_{SK[G]} = [0]_{SK[G]}$, $\chi_\sigma^G(T(\varphi\psi^{-1})_G)) = 0$. Additionally, we also have $\chi_\sigma^G(M) = \chi_\sigma^G(M')$ so the above simplifies to $\chi_\sigma^G(2Y) = 2\chi_\sigma^G(Y) - 2\chi_\sigma^G(M')$. As noted above $2Y$ is a closed $n+1$-dimensional manifold, and so $\chi_\sigma^G(2Y) = 0$. This gives $0 = 2\chi_\sigma^G(Y) - 2\chi_\sigma^G(M') \Rightarrow \chi_\sigma^G(Y) = \chi_\sigma^G(M')$ □

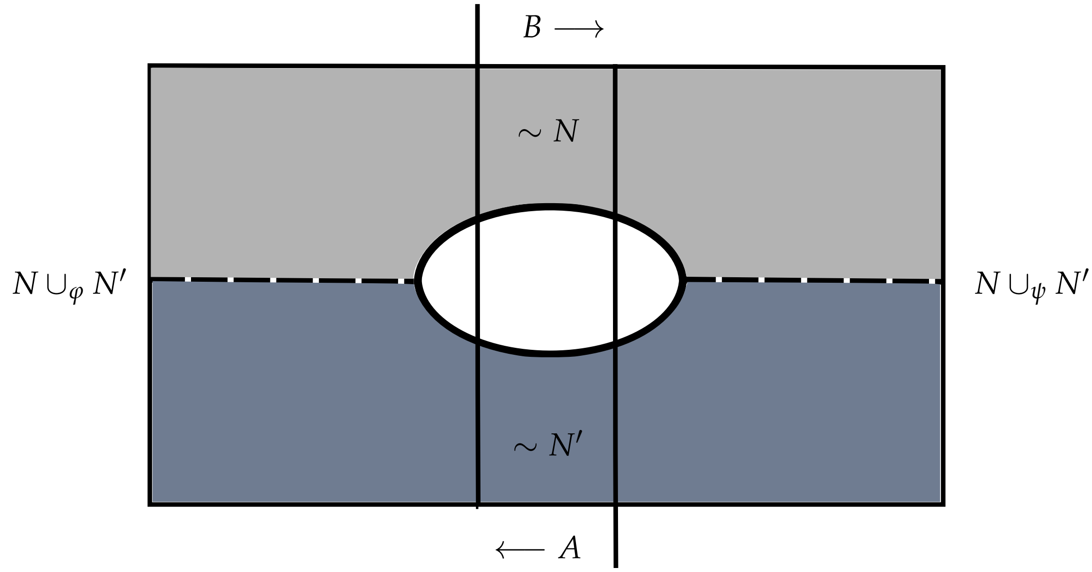

FIGURE 8. Partitioning the bordism of SK Equivalent Manifolds

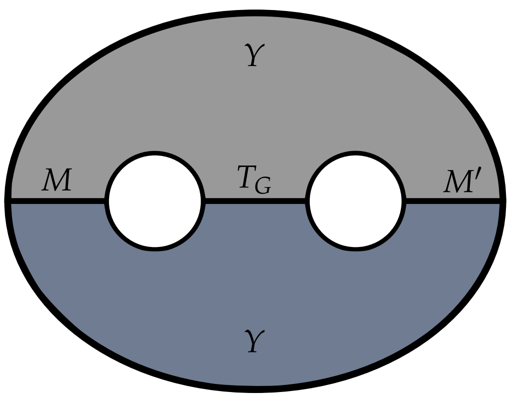

FIGURE 9. The Double of a Cobordism of SK-equivalent Manifolds

Lemma 5.2. *$I_n[G] = I'_n[G]$, where*

$$I_n[G] = \{[G \times_H S(V_i)]_{SK[G]} | \text{ All Representation } V_i\}$$

$$I'_n[G] = \{[G \times_H S(V_i)]_{SKK[G]} | \text{ All Representations } V_i\}$$

Proof. First note that this map is a surjection, as assigning $SK[G]$ class is a quotient, and both groups are generated by the same representatives in $\mathcal{M}_n[G]$. Now, we must show the map is injective. Take two classes of $I'_n[G]$, $[S_1]_{SKK[G]}$ and $[S_2]_{SKK[G]}$, such that $[S_1]_{SK[G]} = [S_2]_{SK[G]}$. Now, if we look at the equivariant Euler characteristic of both manifolds, by the prior lemma $\chi^G_\sigma(S_1) = \chi^G_\sigma(S_2) = \chi^G_\sigma(Y)$. Secondly, as $[S_1]_{SKK}$ and $[S_2]_{SKK}$ are both in $I'_n[G]$, and so by the exact sequence 4.3, they both take the zero class in $\Omega_n[G]$. Then we may use [WW88, Lemma 4.2 and Theorem 1], which states that if $[M]_{\Omega_n[G]} = [M]_{\Omega N_n[G]}$ and $\chi^G_\sigma(M) = \chi^G_\sigma(M') = \chi^G_\sigma(Y)$, where Y is the bordism between M and M' then $[M]_{SKK[G]} = [M']_{SKK[G]}$. So now we see that if $[S_1]_{SK[G]} = [S_2]_{SK[G]}$ and $S_1, S_2 \in I'_n[G]$, then $[S_1]_{SKK[G]} = [S_2]_{SKK[G]}$, which proves injectivity, and so we have an isomorphism. ☐

Using the above proofs and mimicking the proof of 2.9, we arrive at the following theorem.

Theorem 5.2. *The following sequence is for n even in the oriented case and for all n in the non-oriented case.*

$$0 \longrightarrow F_n[G] \longrightarrow SKK_n[G] \longrightarrow SK_n[G] \longrightarrow 0 \tag{5.4}$$

As in the non-equivariant case, this is a consequence of the below braid commuting for n even in the oriented case and all n in the non-oriented case.

$$\begin{tikzcd}
 & I_n[G] \arrow[rr, bend left] \arrow[dr] & & SK_n[G] \arrow[rr, bend left] \arrow[dr] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & SKK_n[G] \arrow[ur] \arrow[dr] & & \overline{SK}_n[G] \arrow[ur] \arrow[dr] & & 0 \\
 & F_n[G] \arrow[ur] \arrow[rr, bend right] & & \Omega^\bullet_n[G] \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] &
\end{tikzcd} \tag{5.5}$$

Note that once again, we encounter issues of torsion for $n \equiv 1 \pmod 4$. Additionally, the $n \equiv 3 \pmod 4$ case has become more interesting, by the introduction of torsion in this dimension. In the context of $SKV[G]$, we can now present an analogous result in all dimensions.

Theorem 5.3. *The below sequence is exact for both oriented and non-oriented manifolds and all n.*

$$0 \longrightarrow F_n[G] \longrightarrow SKV_n[G] \longrightarrow SK_n[G] \longrightarrow 0 \tag{5.6}$$

We see this is exact once again by noticing that the below braid commutes.

$$
\begin{tikzcd}
 & I_n[G] \arrow[dr] \arrow[rr, bend left] & & SK_n[G] \arrow[dr] \arrow[rr, bend left] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & SKV_n[G] \arrow[ur] \arrow[dr] & & \overline{SK}_n[G] \arrow[ur] \arrow[dr] & & 0 \\
 & F_n[G] \arrow[ur] \arrow[rr, bend right] & & \Omega^{\bullet}_n[G] \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] &
\end{tikzcd} \tag{5.7}
$$

It is also worthy of note that the notions of nice equivariant bordism and equivariant bordism (without the fixed point condition) agree for even dimension, but do not agree for odd dimension, which suggests future work to study.

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VOLUMES OF SPHERICAL AND HYPERBOLIC SIMPLICES

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Abstract. This report concerns the volume of spherical and hyperbolic simplices. We will first give some brief introduction to the general background for this report. We then present two simple findings, one on a restatement of an iterated integral formula for n-simplices by Aomoto and another on an alternate proof of a determinant formula for 2-simplices by Tuynman. Then we will present some developments on volume power series. Lastly we will discuss some work on the volume of ideal hyperbolic simplices and future direction.

Contents

1. Introduction

We study the volumes of n-simplices in the spherical and hyperbolic spaces $\mathbb{S}^n$ and $\mathbb{H}^n$, which are of constant curvature $\kappa = \pm 1$.

The Space: As sets, we have

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\}$$

and

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n,1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\},$$

so that $\mathbb{S}^n$ is the surface of a sphere in $(n+1)$-space and $\mathbb{H}^n$ is the upper-half sheet of a hyperboloid in $(n+1)$-space[1]. Geometrically, these two sets will inherit a metric structure from the ambient space; for example, the volume form on $\mathbb{S}^n$ is the volume form on $\mathbb{R}^{n+1}$ restricted to the surface of the sphere. We equip $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with

$$\langle x; y\rangle := x_1y_1 + \cdots + x_ny_n + x_{n+1}y_{n+1} = x^\dagger \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix} y$$

(which we may sometimes refer to as $\langle\cdot;\cdot\rangle_+$) and $\mathbb{H}^n \subset \mathbb{R}^{n+1}$ with

$$\langle x; y\rangle_- := x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1} = x^\dagger \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} y.$$

We will denote this bilinear form matrix by

$$Q = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \kappa \end{pmatrix}.$$

Recalling that $\omega_M = \sum_i u_i * \mathrm{d}x_i$, where u is the vector (plus-or-minus-)unit and normal to M with respect to the appropriate bilinear form, note that

$$\omega_{\text{vol}} := \frac{1}{\|x\|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1} x_i \,\mathrm{d}x_1 \wedge \cdots \wedge \cancel{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{n+1}$$

is the scale-invariant volume form on both $\mathbb{S}^n$ and $\mathbb{H}^n$.

It is a fact that geodesics on these two spaces are traced out by their intersections with hyperplanes passing through the origin (i.e. hyperplanes defined by linear forms). In particular, the distance between two points on these two spaces is given by

$$\begin{aligned} \cos d(p,q) &= \langle p; q\rangle && \text{on } \mathbb{S}^n, \\ -\cosh d(p,q) &= \langle p; q\rangle_- && \text{on } \mathbb{H}^n. \end{aligned}$$

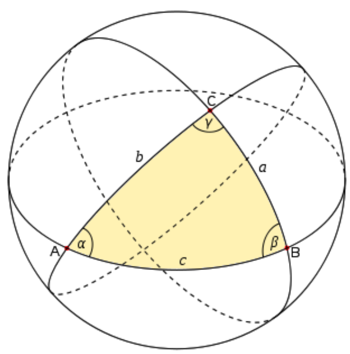
[1]There are many other models for hyperbolic space, but this one will be most extensively used. The Klein projective model will be very briefly touched upon later.

The Simplex: An n-simplex is defined to be the convex hull of $n+1$ points. To avoid degenerate cases, we require that not all of these $n+1$ points lie on a totally geodesic submanifold (i.e. a copy of $\mathbb{S}^{n-1}$ or $\mathbb{H}^{n-1}$, up to scaling in the latter case). The n-simplex can alternatively be formulated as the intersection of a "cone" with the space (either $\mathbb{S}^n$, $\mathbb{H}^n$, or the Euclidean n-space

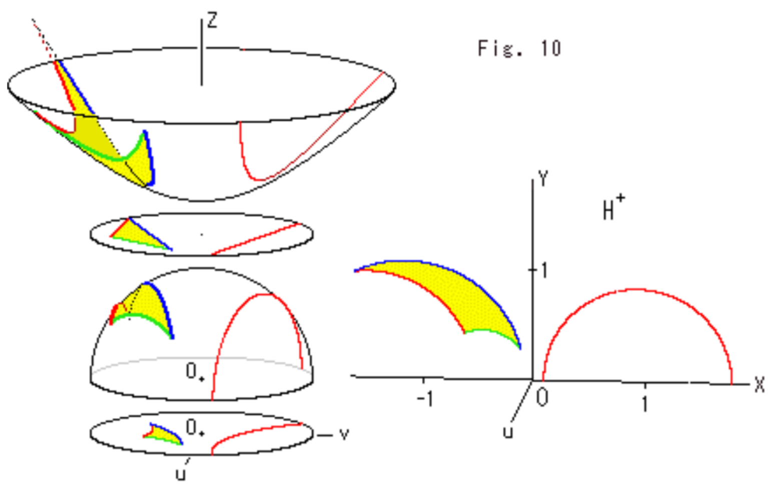
$$\mathbb{E}^n \coloneqq \{x \in \mathbb{R}^{n+1} : x_{n+1} = 1\}$$

sitting inside $\mathbb{R}^{n+1}$) as follows: Let $F_1, \cdots, F_{n+1}$ be linear forms (such that the hyperplanes they define are distinct) which define the spaces $H_i = \{x : F_i(x) \geq 0\}$. The intersection of these spaces $\cap_{i=1}^{n+1} H_i$ forms a cone C. The intersection $C \cap \mathbb{S}^n$ or $\mathbb{H}^n$ or $\mathbb{E}^n$ will give, respectively, a spherical or hyperbolic or Euclidean simplex.

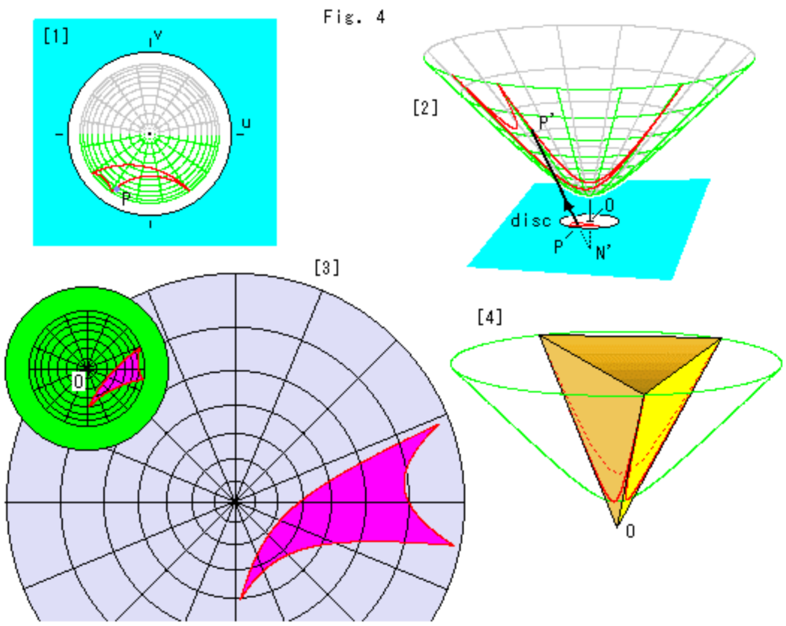
We now give some examples of spherical and hyperbolic 2-simplices. Note that the geodesics are, in fact, intersections with hyperplanes.



A spherical 2-simplex.



A hyperbolic 2-simplex in many different models, most notably the hyperboloid model in the top left, which we focus on.



Note the intersection of the cone with the hyperboloid in the bottom right.

Constructions: Let Δ be an n-simplex determined by the vertices $p_1, \cdots, p_{n+1}$. We construct the (inward) normal unit vectors $\{u_i\}_i$ as follows: for each i, consider all the vertices except p_i. These n points determine a hyperplane passing through the origin, and this hyperplane has a unique unit normal (with respect to the appropriate bilinear form) vector pointing into the simplex. Let this vector be u_i. Note that this construction is dual in the sense that given $\{u_i\}_i$, we can also reconstruct $\{p_i\}_i$.

We can then define

$$P := \begin{pmatrix} \uparrow & \cdots & \uparrow \\ p_1 & \cdots & p_{n+1} \\ \downarrow & \cdots & \downarrow \end{pmatrix}, \qquad U := \begin{pmatrix} \uparrow & \cdots & \uparrow \\ u_1 & \cdots & u_{n+1} \\ \downarrow & \cdots & \downarrow \end{pmatrix},$$

which we use to define

$$G := U^\dagger Q U, \qquad L := P^\dagger Q P,$$

the *Gram matrix* and the *length matrix* respectively. Note that both G and L are symmetric. The entries of G are $\langle u_i; u_j \rangle_Q$ while the entries of L are $\langle p_i; p_j \rangle_Q$.

We then define the dihedral angle between two faces F_i and F_j to be $\theta_{ij} = \theta_{ji} := \pi - \operatorname{ang}(u_i, u_j)$, so that G is of form

$$G = \begin{pmatrix} 1 & & -\cos\theta_{ij} \\ & \ddots & \\ -\cos\theta_{ji} & & 1 \end{pmatrix}.$$

Note that the choice $\theta_{ii} := \pi$ is consistent with the unit length of the normal vectors[2].

[2]It should perhaps be noted here that in the hyperbolic case, although $\langle p_i; p_i \rangle_- = -1$, the u_i's actually lie on the de Sitter sphere, which gives $\langle u_i; u_i \rangle_- = 1$. So there is more going on beneath the surface here than it would appear, and for a more careful treatment we refer the reader to Peiro's treatment. Perhaps it is better that we speak less of that here.

Due to the distance formulas stated earlier, we have that L is of form

$$L = \begin{pmatrix} 1 & & \cos l_{ij} \\ & \ddots & \\ \cos l_{ji} & & 1 \end{pmatrix} \qquad \text{in } \mathbb{S}^n,$$

$$L = \begin{pmatrix} -1 & & -\cosh l_{ij} \\ & \ddots & \\ -\cosh l_{ji} & & -1 \end{pmatrix} \qquad \text{in } \mathbb{H}^n,$$

where $l_{ij} = l_{ji}$ is the edge length between the vertices labeled by i and j. In particular note

$$\kappa L = \begin{pmatrix} 1 & & \cos(\sqrt{\kappa}\, l_{ij}) \\ & \ddots & \\ \cos(\sqrt{\kappa}\, l_{ji}) & & 1 \end{pmatrix}.$$

Before giving further properties of these constructs, let us quickly define some matrix notation. Let $M \in \mathrm{Mat}_{n\times n}$. For $I, J \overset{\text{Set}}{\subseteq} [n]$ define M^I_J to be the matrix given by taking the rows indexed by I and the columns indexed by J, and similarly let $M^{\setminus I}_{\setminus J}$ be the matrix given by taking the rows indexed by $[n] \setminus I$ and the columns indexed by $[n] \setminus J$. We will abbreviate singleton notation, so that $M^{\setminus i}_{\setminus j}$ refers to $M^{\setminus\{i\}}_{\setminus\{j\}}$. We abbreviate $M^{I\cup\{a\}}_{J\cup\{b\}}$ to $M^{I,a}_{J,b}$.

One can imagine that not all symmetric unidiagonal[3] matrices correspond to simplices (c.f. triangle inequality). However, Luo and Milnor found necessary and sufficient conditions for Gram matrices to correspond to simplices.

Theorem (Luo)**.**
In $\mathbb{S}^n$, given a set of angles $\{\theta_{ii} = \pi,\ \theta_{ij} \in (0, \pi)\}_{i,j=1}^{n+1}$, there is a spherical simplex Δ with those dihedral angles if and only if $G = \{-\cos\theta_{ij}\}_{i,j=1}^{n+1}$ is: (i) positive definite.

In $\mathbb{H}^n$, given a set of angles $\{\theta_{ii} = \pi,\ \theta_{ij} \in (0, \pi)\}_{i,j=1}^{n+1}$, there is a hyperbolic simplex Δ with those dihedral angles if and only if $G = \{-\cos\theta_{ij}\}_{i,j=1}^{n+1}$ has: (i) determinant $\det G < 0$, (ii) all principal submatrices positive definite, and (iii) all i,j-th cofactors $(-1)^{i+j}\det G^{\setminus i}_{\setminus j} > 0$.

In $\mathbb{E}^n$, given a set of angles $\{\theta_{ii} = \pi,\ \theta_{ij} \in (0, \pi)\}_{i,j=1}^{n+1}$, there is a Euclidean simplex Δ with those dihedral angles if and only if $G = \{-\cos\theta_{ij}\}_{i,j=1}^{n+1}$ has: (i) determinant $\det G = 0$, (ii) all principal submatrices positive definite, and (iii) all i,j-th cofactors $(-1)^{i+j}\det G^{\setminus i}_{\setminus j} > 0$.

The last result is perhaps less relevant than the first two, but we include it for completeness. Note that we then have $\operatorname{sgn}\det G = \det Q = \kappa$.

Define

$$T := \{\langle p_i; u_j\rangle_Q\}_{i,j=1}^{n+1};$$

we then have the following two facts, which are well-known in the field and for example noted in a paper by Murakami and Ushijima.

[3] Meaning having all diagonal entries of 1.

Theorem. In $\mathbb{S}^n$ and $\mathbb{H}^n$,

$$G = TL^{-1}T, \qquad L = TG^{-1}T.$$

Proof. Recall $i \neq j \implies \langle u_i; p_j \rangle = 0$ by construction, so T is diagonal. Then

$$U^\dagger Q P = \begin{pmatrix} \leftarrow & u_1 & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & u_{n+1} & \rightarrow \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \kappa \end{pmatrix} \begin{pmatrix} \uparrow & \cdots & \uparrow \\ p_1 & \cdots & p_{n+1} \\ \downarrow & \cdots & \downarrow \end{pmatrix} = \begin{pmatrix} \langle u_1; p_1 \rangle_Q & & \\ & \ddots & \\ & & \langle u_{n+1}; p_{n+1} \rangle_Q \end{pmatrix} = T$$

$$\begin{aligned} &\implies U^\dagger Q P = T \\ &\implies QP = (U^\dagger)^{-1}T, \ P^\dagger Q = T^\dagger U^{-1} = TU^{-1} \\ &\implies QP = (U^\dagger)^{-1}T, \ P^\dagger = TU^{-1}Q \\ &\implies P^\dagger Q P = TU^{-1}Q(U^\dagger)^{-1}T = T(U^\dagger Q U)^{-1}T \\ &\implies L = TG^{-1}T, \end{aligned}$$

which proves the claim, as $L = TG^{-1}T \iff T^{-1}L = G^{-1}T \iff L^{-1}T = T^{-1}G \iff TL^{-1}T = G$. ■

Note that the above theorem implies $\operatorname{sgn}\det G = \operatorname{sgn}\det L = \kappa$, provided $\det T \neq 0$. The matrix T is characterized as follows:

Theorem. In fact,

$$\langle p_i; u_i \rangle_Q = \kappa \cdot \sqrt{\frac{\kappa \det G}{\det G^{\setminus i}_{\setminus i}}} = \sqrt{\frac{\det L}{\det L^{\setminus i}_{\setminus i}}},$$

so moreover

$$T = \kappa \begin{pmatrix} \sqrt{\frac{\kappa \det G}{\det G^{\setminus 1}_{\setminus 1}}} & & \\ & \ddots & \\ & & \sqrt{\frac{\kappa \det G}{\det G^{\setminus n+1}_{\setminus n+1}}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\det L}{\det L^{\setminus 1}_{\setminus 1}}} & & \\ & \ddots & \\ & & \sqrt{\frac{\det L}{\det L^{\setminus n+1}_{\setminus n+1}}} \end{pmatrix}.$$

We also have

$$L^i_j = \kappa \cos\big(\sqrt{\kappa}\, l_{ij}\big) = \kappa \frac{(-1)^{i+j}\det G^{\setminus i}_{\setminus j}}{\sqrt{\det G^{\setminus i}_{\setminus i} \det G^{\setminus j}_{\setminus j}}}, \qquad G^i_j = -\cos\theta_{ij} = \kappa \frac{(-1)^{i+j}\det L^{\setminus i}_{\setminus j}}{\sqrt{\det L^{\setminus i}_{\setminus i} \det L^{\setminus j}_{\setminus j}}}.$$

Proof. [4]Before starting, we state two matrix identities which will be helpful (and which will be stated again later for completion):

$$\delta_{ij} \det M = \sum_{\mu=1}^{n} (-1)^{i+\mu} M^j_\mu \det M^{\setminus i}_{\setminus \mu} = \sum_{\mu=1}^{n} (-1)^{i+\mu} M^\mu_j \det M^{\setminus \mu}_{\setminus i},$$

[4]Before starting the "before starting", perhaps it should be noted that I was unable to find a proof (or an entirely precise statement) of the above fact and had to craft an argument myself. Clearly this is a fact well-known in the field (and in fact stated in different forms in several different papers), but in my opinion the proof is perhaps not entirely obvious and lacking from the literature.

$$\det(M^{-1})^I_J = (-1)^{|I|_1+|J|_1} \frac{\det M^{\setminus J}_{\setminus I}}{\det M},$$

where $|I|_1 := \sum_{i \in I} i$ (as opposed to $|I|_0 := |I| = \sum_{i \in I} 1$).

There are four equalities to prove. Recall the duality of p and u.

Given $\{u_i\}$, consider

$$v_i := \frac{\sum_{j=1}^{n+1} (-1)^{i+j} \det G^{\setminus i}_{\setminus j} u_j}{\sqrt{|\det G| \cdot \det G^{\setminus i}_{\setminus i}}}.$$

Observe that by Luo the denominator is real. This $\{v_i\}$ has two properties: that

$$\begin{aligned}
\langle v_i; v_i \rangle_Q &= \frac{1}{\det G^{\setminus i}_{\setminus i} |\det G|} \sum_{j,k \in [n+1]}\sum (-1)^{j+k} \det G^{\setminus i}_{\setminus j} \det G^{\setminus i}_{\setminus k} \langle u_j; u_k \rangle_Q \\
&= \frac{1}{\det G^{\setminus i}_{\setminus i} |\det G|} \sum_{j=1}^{n+1} (-1)^{j+i} \det G^{\setminus i}_{\setminus j} \sum_{k=1}^{n+1} (-1)^{i+k} G^j_k \det G^{\setminus i}_{\setminus k} \\
&= \frac{1}{\det G^{\setminus i}_{\setminus i} |\det G|} \sum_{j=1}^{n+1} (-1)^{j+i} \det G^{\setminus i}_{\setminus j} \delta_{ij} \det G \\
&= \frac{1}{\det G^{\setminus i}_{\setminus i} |\det G|} \det G^{\setminus i}_{\setminus i} \det G \\
&= \operatorname{sgn} \det G \\
&= \kappa,
\end{aligned}$$

so it is of (plus-or-minus-)unit length, and that $i \neq j \implies$

$$\begin{aligned}
\langle u_i; v_j \rangle_Q &= \sum_{k=1}^{n+1} \frac{(-1)^{j+k} \det G^{\setminus j}_{\setminus k}}{\sqrt{\det G^{\setminus j}_{\setminus j} |\det G|}} \langle u_i; u_k \rangle_Q \\
&= \frac{1}{\sqrt{\det G^{\setminus j}_{\setminus j} |\det G|}} \sum_{k=1}^{n+1} (-1)^{j+k} G^i_k \det G^{\setminus j}_{\setminus k} \\
&= \frac{1}{\sqrt{\det G^{\setminus j}_{\setminus j} |\det G|}} \delta_{ij} \det G \\
&= 0.
\end{aligned}$$

These two properties imply that $\{v_i\} = \{p_i\}$, and the last calculation above incidentally gives

$$\langle u_i; p_i \rangle_Q = \kappa \cdot \sqrt{\frac{\kappa \det G}{\det G^{\setminus i}_{\setminus i}}},$$

which proves one equality. In addition, note that since $L = TG^{-1}T$, we have

$$\begin{aligned}
L^i_j &= \sum_{\mu,\nu}\sum T^i_\mu (G^{-1})^\mu_\nu T^\nu_j \\
&= \sum_{\mu,\nu}\sum \delta_{i\mu}\langle p_i; u_\mu\rangle_Q \frac{(-1)^{\mu+\nu}\det G^{\setminus\mu}_{\setminus\nu}}{\det G}\delta_{\nu j}\langle p_\nu; u_j\rangle_Q \\
&= \langle p_i; u_i\rangle_Q \frac{(-1)^{i+j}\det G^{\setminus i}_{\setminus j}}{\det G}\langle p_j; u_j\rangle_Q \\
&= \kappa\cdot\sqrt{\frac{\kappa\det G}{\det G^{\setminus i}_{\setminus i}}}\,\frac{(-1)^{i+j}\det G^{\setminus i}_{\setminus j}}{\det G}\,\kappa\cdot\sqrt{\frac{\kappa\det G}{\det G^{\setminus j}_{\setminus j}}} \\
&= \kappa\,\frac{(-1)^{i+j}\det G^{\setminus i}_{\setminus j}}{\sqrt{\det G^{\setminus i}_{\setminus i}\det G^{\setminus j}_{\setminus j}}},
\end{aligned}$$

which proves another equality. It remains to prove two more.

This time, given $\{p_i\}$, consider

$$v_i := \frac{\sum_{j=1}^{n+1}(-1)^{i+j}\det L^{\setminus i}_{\setminus j}p_j}{\sqrt{\det L\cdot\det L^{\setminus i}_{\setminus i}}}.$$

First we must justify the denominator being real. Note $G = TL^{-1}T \implies G^i_i = \sum_\mu\sum_\nu T^i_\mu(L^{-1})^\mu_\nu T^\nu_i = T^i_i(L^{-1})^i_i T^i_i = (T^i_i)^2(L^{-1})^i_i$; but $G^i_i = 1 > 0$, which implies $(L^{-1})^i_i > 0 \implies \det L\cdot\det L^{\setminus i}_{\setminus i} = \det L\cdot(-1)^{2|\setminus i|_1}\frac{\det(L^{-1})^i_i}{\det L^{-1}} = (\det L)^2(L^{-1})^i_i > 0 \implies \det L\cdot\det L^{\setminus i}_{\setminus i} > 0$. So the denominator is in fact the square root of a positive number, which is real. This $\{v_i\}$ has again two properties: that

$$\begin{aligned}
\langle v_i; v_i\rangle_Q &= \frac{1}{\det L^{\setminus i}_{\setminus i}\cdot\det L}\sum_{j,k\in[n+1]}\sum(-1)^{j+k}\det L^{\setminus i}_{\setminus j}\det L^{\setminus i}_{\setminus k}\langle p_j; p_k\rangle_Q \\
&= \frac{1}{\det L^{\setminus i}_{\setminus i}\cdot\det L}\sum_{j=1}^{n+1}(-1)^{j+i}\det L^{\setminus i}_{\setminus j}\sum_{k=1}^{n+1}(-1)^{i+k}L^j_k\det L^{\setminus i}_{\setminus k} \\
&= \frac{1}{\det L^{\setminus i}_{\setminus i}\cdot\det L}\sum_{j=1}^{n+1}(-1)^{j+i}\det L^{\setminus i}_{\setminus j}\delta_{ij}\det L \\
&= \frac{1}{\det L^{\setminus i}_{\setminus i}\cdot\det L}\;\det L^{\setminus i}_{\setminus i}\det L \\
&= 1,
\end{aligned}$$

so it is of unit length[5], and that $i \neq j \implies$

$$\begin{aligned}
\langle p_i; v_j \rangle_Q &= \sum_{k=1}^{n+1} \frac{(-1)^{j+k} \det L^{\setminus j}_{\setminus k}}{\sqrt{\det L^{\setminus j}_{\setminus j} \cdot \det L}} \langle p_i; p_k \rangle_Q \\
&= \frac{1}{\sqrt{\det L^{\setminus j}_{\setminus j} \cdot \det L}} \sum_{k=1}^{n+1} (-1)^{j+k} L^i_k \det L^{\setminus j}_{\setminus k} \\
&= \frac{1}{\sqrt{\det L^{\setminus j}_{\setminus j} \cdot \det L}} \delta_{ij} \det L \\
&= 0.
\end{aligned}$$

These two properties imply that $\{v_i\} = \{u_i\}$, and the last calculation above incidentally gives

$$\langle p_i; u_i \rangle_Q = \sqrt{\frac{\det L}{\det L^{\setminus i}_{\setminus i}}},$$

which proves another equality. In addition, note that since $G = TL^{-1}T$, we have

$$\begin{aligned}
G^i_j &= \sum_{\mu,\nu} \sum T^i_\mu (L^{-1})^\mu_\nu T^\nu_j \\
&= \sum_{\mu,\nu} \sum \delta_{i\mu} \langle p_i; u_\mu \rangle_Q \frac{(-1)^{\mu+\nu} \det L^{\setminus \mu}_{\setminus \nu}}{\det L} \delta_{\nu j} \langle p_\nu; u_j \rangle_Q \\
&= \langle p_i; u_i \rangle_Q \frac{(-1)^{i+j} \det L^{\setminus i}_{\setminus j}}{\det L} \langle p_j; u_j \rangle_Q \\
&= \sqrt{\frac{\det L}{\det L^{\setminus i}_{\setminus i}}} \frac{(-1)^{i+j} \det L^{\setminus i}_{\setminus j}}{\det L} \sqrt{\frac{\det L}{\det L^{\setminus j}_{\setminus j}}} \\
&= \kappa \frac{(-1)^{i+j} \det L^{\setminus i}_{\setminus j}}{\sqrt{\det L^{\setminus i}_{\setminus i} \det L^{\setminus j}_{\setminus j}}},
\end{aligned}$$

which shows the last equality. This finishes the proof. ■

[5]This is strictly positive one and never negative, which again reflects the fact that in the hyperbolic case the unit normal vectors lie on the de Sitter sphere. Of course in the spherical case it is of positive length. But enough of that.

2. On an Iterated Integral Formula

We investigate an iterated integral formula for the volume of a simplex given by Aomoto and modified by Kohno. For a more thorough exposition we refer the reader to Kohno's (perhaps dense) survey.

Previous Work: First we give some notation. For a simplex Δ, let its faces be labeled as $F_1, \cdots, F_{n+1}$. Let $I_1 \subset \cdots \subset I_{\lfloor n/2 \rfloor}$ be a sequence of index sets of size $|I_k| = 2k$. For I_1, we define θ_{I_1} to be the angle between the two faces labeled by I_1, and for each $I_{k+1} = I_k \cup \{a, b\}$ we define $\theta_{I_k \to I_{k+1}}$ to be the dihedral angle between $F_a \cap \bigcap_{i \in I_k} F_i$ and $F_b \cap \bigcap_{i \in I_k} F_i$:

$$\theta_{I_k \to I_{k+1}} \coloneqq \operatorname{ang}\left(F_a \cap \bigcap_{i \in I_k} F_i, F_b \cap \bigcap_{i \in I_k} F_i \right).$$

When the context is clear we may suppress $\theta_{I_k \to I_{k+1}}$ to $\theta_{I_{k+1}}$; that is, we let I_{k+1} contain all the information about its predecessors. We'll also denote $\Delta_{I_k} \coloneqq \bigcap_{i \in I_k} F_i$.

We can then think of θ_{I_k} as a function on the space of Gram matrices, or the space of "configurations", so to speak. We recall a result from Aomoto (and rewritten by Kohno) which expresses the differential of this function:

Theorem (Aomoto)**.**

$$\mathrm{d}\theta_{I \to I \cup \{a,b\}}(G) = \mathrm{d}\tan^{-1}\left(-\frac{\sqrt{\det G_I^I \det G_{I,a,b}^{I,a,b}}}{\det G_{I,b}^{I,a}} \right).$$

Recall the Schlafli differential equality:

Theorem (Schlafli)**.**

$$\kappa \, \mathrm{dvol}_n(\Delta) = \frac{1}{n-1} \sum_{I_1} \mathrm{vol}_{n-2} \Delta_{I_1} \, \mathrm{d}\theta_{I_1}.$$

Kohno then uses the above to give the following theorem, which was first stated for odd n by Aomoto. Before stating it, recall that the iterated integral $\int_{\mathrm{Id} \to M} \omega_k \cdots \omega_1$ for 1-forms ω_i on the space Γ of matrix variables γ is defined as follows: First we integrate $\int_{\mathrm{Id} \to M} \omega_k(\gamma)$ to get a function $f_k(M)$ on Γ of the endpoint M, and then we integrate $\int_{\mathrm{Id} \to M} f_k(\gamma)\omega_{k-1}(\gamma)$ to get another function $f_{k-1}(M)$ of the endpoint, and so forth and so on, until we reach a function $f_1(M)$, which is then defined to be the value of the iterated integral. Note that the iterated integral is a function of its endpoint. Recall $\mathrm{vol}_n(\mathbb{S}^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$ and $(-1)!! = 1$.

Theorem (Aomoto, Kohno).

$$\kappa^{\frac{n}{2}} \operatorname{vol}_n(\Delta(G)) = c_{n,0} + \sum_{k=1}^{n/2} c_{n,k} \sum_{I_k \supset \cdots \supset I_1} \int_{\mathrm{Id} \to G} \mathrm{d}\theta_{I_k} \cdots \mathrm{d}\theta_{I_1},$$

where

$$c_{n,k} = \frac{(n-1-2k)!!}{(n-1)!!} \frac{\operatorname{vol}_{n-2k}(\mathbb{S}^{n-2k})}{2^{n+1-2k}}.$$

If $\frac{n}{2} \notin \mathbb{Z}$, the last term is not immediately well-defined, so we clarify as thus: the sum is defined to have $\lfloor \frac{n}{2} \rfloor = m$ many components, i.e. for $k = 1, 2, \cdots, m-1, m+0.5$. For $k = m + 0.5$ the summand is defined to have integrand

$$\operatorname{vol}_1(\Delta_{I_m})\, \mathrm{d}\theta_{I_m} \cdots \mathrm{d}\theta_{I_1},$$

and the chain of index sets summed over is $I_m \supset \cdots \supset I_1$.

Proof. To give a full proof here seems too ambitious, and in any case we refer the reader to Kohno's paper for the full (perhaps flawed?) proof. Kohno states the above formula in a slightly different form, which I am fairly certain is not entirely correct. The rough idea is to iterate using Schlafli's formula in the spherical case and then extend to the hyperbolic case by using the analyticity of the volume-corrected curvature, $\kappa(\operatorname{vol}_n \Delta)^{\frac{2}{n}}$. To give an example of the first steps of the proof in the spherical case,

$$\begin{aligned}
\mathrm{d}\operatorname{vol}_n \Delta(G) &= \frac{1}{n-1} \sum_{I_1} \operatorname{vol}_{n-2} \Delta_{I_1} \mathrm{d}\theta_{I_1} \\
\int_{\mathrm{Id} \to G} \mathrm{d}\operatorname{vol}_n \Delta(\gamma) &= \int_{\mathrm{Id} \to G} \frac{1}{n-1} \sum_{I_1} \operatorname{vol}_{n-2} \Delta_{I_1} \mathrm{d}\theta_{I_1} \\
\operatorname{vol}_n \Delta(G) - \operatorname{vol}_n \Delta(\mathrm{Id}) &= \frac{1}{n-1} \sum_{I_1} \int_{\mathrm{Id} \to G} \operatorname{vol}_{n-2} \Delta_{I_1} \mathrm{d}\theta_{I_1} \\
\operatorname{vol}_n \Delta(G) &= \frac{\operatorname{vol}_n \mathbb{S}^n}{2^{n+1}} + \frac{1}{n-1} \sum_{I_1} \int_{\mathrm{Id} \to G} \operatorname{vol}_{n-2} \Delta_{I_1}(\gamma)\, \mathrm{d}\theta_{I_1}(\gamma);
\end{aligned}$$

however, we can also use the Schlafli equality on vol_{n-2} to obtain

$$\operatorname{vol}_{n-2} \Delta_{I_1}(G) = \frac{\operatorname{vol}_{n-2} \mathbb{S}^{n-2}}{2^{n-1}} + \frac{1}{n-3} \sum_{I_2 : I_2 \supset I_1} \int_{\mathrm{Id} \to G} \operatorname{vol}_{n-4} \Delta_{I_2}(\gamma)\, \mathrm{d}\theta_{I_2}(\gamma),$$

and so on for each $\operatorname{vol}_{n-2k}$. After substituting all these we obtain the desired result. It is important to observe that, for odd n, this sequence of steps of expressing smaller $\operatorname{vol}_{n-2k}$ terminates with a vol_1 term, which cannot be broken down further by Schlafli; hence the sum defined up to $n/2$ for odd n in the statement above.

Aomoto and Kohno then each use an analytic continuation argument to extend this to work in the hyperbolic case. For this step the reader may peruse their papers. ■

Matrix Identities: Next we recall some facts from linear algebra.

Theorem (Jacobi)**.**

$$\det M \det M^{\setminus ij}_{\setminus ij} = \det M^{\setminus i}_{\setminus i} \det M^{\setminus j}_{\setminus j} - \det M^{\setminus i}_{\setminus j} \det M^{\setminus j}_{\setminus i}.$$

Note that in the case of symmetric M the above becomes $\det M \det M^{\setminus ij}_{\setminus ij} = \det M^{\setminus i}_{\setminus i} \det M^{\setminus j}_{\setminus j} - \big(\det M^{\setminus i}_{\setminus j}\big)^2$.

Theorem (Laplace)**.**

$$\delta_{ij} \det M = \sum_{\mu=1}^{n} (-1)^{i+\mu} M^j_\mu \det M^{\setminus i}_{\setminus \mu} = \sum_{\mu=1}^{n} (-1)^{i+\mu} M^\mu_j \det M^{\setminus \mu}_{\setminus i}.$$

Theorem (Generalized Laplace)**.** Fix $I \subseteq [n]$. Then

$$\det M = \sum_{J:|J|=|I|} (-1)^{\sum_{i\in I} i + \sum_{j \in J} j} \det M^I_J \det M^{\setminus I}_{\setminus J}.$$

Theorem (inverse)**.**

$$\det (M^{-1})^I_J = (-1)^{\sum_{i\in I} i + \sum_{j \in J} j} \frac{\det M^{\setminus J}_{\setminus I}}{\det M}.$$

Now let $A, B \in \mathrm{Mat}_{n\times n}$.

Theorem (product)**.** Fix $I, J \subseteq [n]$ with $|I| = |J|$. Then

$$\det (AB)^I_J = \sum_{K:|K|=|I|=|J|} \det A^I_K \det B^K_J.$$

Result: We present our first simple finding.

Theorem. In terms of the length matrix L we have

$$\mathrm{d}\theta_{I\to I\cup\{a,b\}}(L) = \mathrm{d}\tan^{-1}\left((-1)^{a+b+1} \frac{\sqrt{\det L^{\setminus I}_{\setminus I} \det L^{\setminus I,a,b}_{\setminus I,a,b}}}{\det L^{\setminus I,a}_{\setminus I,b}} \right).$$

Proof. We compute. For ease of access recall

$$\mathrm{d}\theta_{I\to I\cup\{a,b\}}(G) = \mathrm{d}\tan^{-1}\left(- \frac{\sqrt{\det G^I_I \det G^{I,a,b}_{I,a,b}}}{\det G^{I,a}_{I,b}} \right).$$

We want to substitute G with L. Compute:

$$\det G_I^I = \det(TL^{-1}T)_I^I = \sum_{J:|J|=|I|} \det(TL^{-1})_J^I \det T_I^J;$$

but since T is diagonal, we have that unless $I = J$ as sets identically, some row/column of zeros will appear in $T_I^J \implies \det T_I^J = 0$, so we force $J = I$ as all other terms vanish;

$$\begin{aligned} &= \det(TL^{-1})_I^I \det T_I^I \\ &= \left(\sum_{J:|J|=|I|} \det T_J^I \det(L^{-1})_I^J \right) \det T_I^I \\ &= \det T_I^I \det(L^{-1})_I^I \det T_I^I; \end{aligned}$$

at this point note $\det T_I^I = \prod_{i \in I} \sqrt{\frac{\det L}{\det L_{\setminus i}^{\setminus i}}}$ from our form for T above and recall $\det(L^{-1})_I^I = (-1)^{2\sum_{i\in I} i} \frac{\det L_{\setminus I}^{\setminus I}}{\det L} = \frac{\det L_{\setminus I}^{\setminus I}}{\det L}$; recall also that $\det L$ and $\det L_{\setminus i}^{\setminus i}$ are of the same sign;

$$\begin{aligned} &= \frac{(\det L)^{|I|}}{\prod_{i\in I} \det L_{\setminus i}^{\setminus i}} \frac{\det L_{\setminus I}^{\setminus I}}{\det L} \\ &= \frac{(\det L)^{|I|-1}}{\prod_{i\in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I}^{\setminus I}. \end{aligned}$$

So

$$\det G_I^I = \frac{(\det L)^{|I|-1}}{\prod_{i\in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I}^{\setminus I}, \tag{$*$}$$

which automatically gives

$$\det G_{I,a,b}^{I,a,b} = \frac{(\det L)^{|I|+1}}{\det L_{\setminus b}^{\setminus a} \cdot \det L_{\setminus b}^{\setminus b} \cdot \prod_{i\in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I,a,b}^{\setminus I,a,b}. \tag{$**$}$$

Next we compute

$$\begin{aligned} \deg G_{I,b}^{I,a} = \det(TL^{-1}T)_{I,b}^{I,a} &= \sum_{J:|J|=|I|+1} \det(TL^{-1})_J^{I,a} \det T_{I,b}^J \\ &= \det(TL^{-1})_{I,b}^{I,a} \det T_{I,b}^{I,b} \\ &= \left(\sum_{J:|J|=|I|+1} \det T_J^{I,a} \det(L^{-1})_{I,b}^J \right) \det T_{I,b}^{I,b} \\ &= \det T_{I,a}^{I,a} \det(L^{-1})_{I,b}^{I,a} \det T_{I,b}^{I,b} \\ &= \frac{1}{\sqrt{\det L_{\setminus a}^{\setminus a} \det L_{\setminus b}^{\setminus b}}} \frac{(\det L)^{|I|+1}}{\prod_{i\in I} \det L_{\setminus i}^{\setminus i}} \cdot (-1)^{a+b+2\sum_{i\in I} i} \frac{\det L_{\setminus I,b}^{\setminus I,a}}{\det L} \end{aligned}$$

$$= (-1)^{a+b} \frac{1}{\sqrt{\det L_{\setminus a}^{\setminus a} \det L_{\setminus b}^{\setminus b}}} \frac{(\det L)^{|I|}}{\prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I,b}^{\setminus I,a},$$

so

$$\det G_{I,b}^{I,a} = (-1)^{a+b} \frac{1}{\sqrt{\det L_{\setminus a}^{\setminus a} \det L_{\setminus b}^{\setminus b}}} \frac{(\det L)^{|I|}}{\prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I,b}^{\setminus I,a}. \qquad (***)$$

We can now combine our three starred results together to find

$$\begin{aligned}
-\frac{\sqrt{\det G_I^I \deg G_{I,a,b}^{I,a,b}}}{\det G_{I,b}^{I,a}} &= -\frac{\sqrt{\frac{(\det L)^{|I|-1}}{\prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I}^{\setminus I} \cdot \frac{(\det L)^{|I|+1}}{\det L_{\setminus b}^{\setminus a} \cdot \det L_{\setminus b}^{\setminus b} \cdot \prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I,a,b}^{\setminus I,a,b}}}{(-1)^{a+b} \frac{1}{\sqrt{\det L_{\setminus a}^{\setminus a} \det L_{\setminus b}^{\setminus b}}} \frac{(\det L)^{|I|}}{\prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I,b}^{\setminus I,a}} \\
&= -\frac{\sqrt{\frac{(\det L)^{2|I|}}{\det L_{\setminus b}^{\setminus a} \cdot \det L_{\setminus b}^{\setminus b} \cdot \left(\prod_{i \in I} \det L_{\setminus i}^{\setminus i}\right)^2} \cdot \det L_{\setminus I}^{\setminus I} \cdot \det L_{\setminus I,a,b}^{\setminus I,a,b}}}{(-1)^{a+b} \frac{1}{\sqrt{\det L_{\setminus a}^{\setminus a} \det L_{\setminus b}^{\setminus b}}} \frac{(\det L)^{|I|}}{\prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \det L_{\setminus I,b}^{\setminus I,a}} \\
&= (-1)^{a+b+1} \frac{\frac{1}{\sqrt{\det L_{\setminus a}^{\setminus a} \det L_{\setminus b}^{\setminus b}}} \frac{(\det L)^{|I|}}{\prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \sqrt{\det L_{\setminus I}^{\setminus I} \cdot \det L_{\setminus I,a,b}^{\setminus I,a,b}}}{\frac{1}{\sqrt{\det L_{\setminus a}^{\setminus a} \det L_{\setminus b}^{\setminus b}}} \frac{(\det L)^{|I|}}{\prod_{i \in I} \det L_{\setminus i}^{\setminus i}} \cdot \det L_{\setminus I,b}^{\setminus I,a}} \\
&= (-1)^{a+b+1} \frac{\sqrt{\det L_{\setminus I}^{\setminus I} \cdot \det L_{\setminus I,a,b}^{\setminus I,a,b}}}{\det L_{\setminus I,b}^{\setminus I,a}},
\end{aligned}$$

as desired. ∎

Since this translation of data does not change the nature of the form $\mathrm{d}\theta$, and since the iterated integral in question is homotopy invariant (for example c.f. Kohno), we have that the same formula holds:

Theorem.

$$\kappa^{\frac{n}{2}} \operatorname{vol}_n(\Delta(L)) = c_{n,0} + \sum_{k=1}^{n/2} c_{n,k} \sum_{I_k \supset \cdots \supset I_1} \int_{\mathrm{Id} \to L} \mathrm{d}\theta_{I_k} \cdots \mathrm{d}\theta_{I_1},$$

where

$$c_{n,k} = \frac{(n-1-2k)!!}{(n-1)!!} \frac{\operatorname{vol}_{n-2k}(\mathbb{S}^{n-2k})}{2^{n+1-2k}}$$

and

$$\mathrm{d}\theta_{I_k}(L) = \mathrm{d}\tan^{-1}\left((-1)^{a+b+1} \frac{\sqrt{\det L^{\setminus I_{k-1}}_{\setminus I_{k-1}} \det L^{\setminus I_{k-1},a,b}_{\setminus I_{k-1},a,b}}}{\det L^{\setminus I_{k-1},a}_{\setminus I_{k-1},b}} \right)$$

for $I_k = I_{k-1} \cup \{a, b\}$. If $\frac{n}{2} \notin \mathbb{Z}$, the last term is not immediately well-defined, so we clarify as thus: the sum is defined to have $\left\lfloor \frac{n}{2} \right\rfloor = m$ many components, i.e. for $k = 1, 2, \cdots, m-1, m+0.5$. For $k = m + 0.5$ the summand is defined to have integrand

$$\operatorname{vol}_1(\Delta_{I_m}) \,\mathrm{d}\theta_{I_m} \cdots \mathrm{d}\theta_{I_1},$$

and the chain of index sets summed over is $I_m \supset \cdots \supset I_1$.

Some remarks on this formula: note that, in terms of the data of L, for odd n the last integrand featuring $\operatorname{vol}_1 \Delta_{I_{\lfloor n/2 \rfloor}}$ is easier to express. Note also that whereas for large I the computation of $\mathrm{d}\theta_I(G)$ involved large determinants, in terms of the length matrix L this trope is flipped; in terms of L, larger I gives smaller determinant computations.

3. Another Proof of Tuynman's Formula

We briefly outline the usage of the relation $G = TL^{-1}T$ to derive Tuynman's formula, which states

Theorem (Tuynman)**.** For triangles $\Delta \in \mathbb{S}^n$ or $\mathbb{H}^n$, the area is given by

$$\sin^2\left(\frac{\operatorname{vol}_2 \Delta}{2}\right) = \frac{|\det L|}{2(1+\kappa\lambda_{1,2})(1+\kappa\lambda_{2,3})(1+\kappa\lambda_{3,1})},$$

where λ_{ij} are the entries of L.

Alternative Proof. Recall the classical formula

$$\operatorname{vol}_2 \Delta = \kappa(\alpha + \beta + \gamma - \pi),$$

which incidentally can be trivially recovered from the iterated integral formula given by Aomoto and Kohno. Recall from the introduction that $-\cos\theta_{ij} = \kappa \frac{(-1)^{i+j}\det L^{\setminus i}_{\setminus j}}{\sqrt{\det L^{\setminus i}_{\setminus i}\det L^{\setminus j}_{\setminus j}}}$, so that $\theta_{ij} = \cos^{-1}\left(\kappa \frac{(-1)^{i+j+1}\det L^{\setminus i}_{\setminus j}}{\sqrt{\det L^{\setminus i}_{\setminus i}\det L^{\setminus j}_{\setminus j}}}\right)$. We can then take the cosine of both sides of the classical formula:

$$\begin{aligned}
\cos\operatorname{vol}_2 \Delta &= \cos(\kappa(\theta_{1,2} + \theta_{2,3} + \theta_{3,1} - \pi)) \\
&= -\cos(\theta_{1,2} + \theta_{2,3} + \theta_{3,1}) \\
&= -\cos(\theta_{1,2})\cos(\theta_{2,3})\cos(\theta_{3,1}) + \sum_{\text{cyc}} \cos(\theta_{1,2})\sin(\theta_{2,3})\sin(\theta_{3,1}) \\
1 - \cos\operatorname{vol}_2 \Delta &= 1 + \cos(\theta_{1,2})\cos(\theta_{2,3})\cos(\theta_{3,1}) - \sum_{\text{cyc}} \cos(\theta_{1,2})\sin(\theta_{2,3})\sin(\theta_{3,1}).
\end{aligned}$$

For what follows, let $d \coloneqq \det L$, and $d_{ij} \coloneqq \det L^{\setminus i}_{\setminus j}$. So

$$\cos\theta_{ij} = \kappa \frac{(-1)^{i+j+1} d_{ij}}{\sqrt{d_{ii}d_{jj}}}.$$

Given $\cos\theta_{ij}$, we can calculate $\sin\theta_{ij} = \frac{\sqrt{d_{ii}d_{jj} - d_{ij}^2}}{\sqrt{d_{ii}d_{jj}}}$. Note that this is positive, as we require $0 < \theta_{ij} < \pi$. Recall Jacobi's identity, which gives $d_{ii}d_{jj} - d_{ij}^2 = d\det L^{\setminus ij}_{\setminus ij} = \kappa d$, where we observe that the diagonal entries of L are κ. This gives

$$\sin\theta_{ij} = \frac{\sqrt{\kappa d}}{\sqrt{d_{ii}d_{jj}}}.$$

We can then compute

$$
\begin{aligned}
1-\cos\operatorname{vol}_2\Delta &= 1+\cos(\theta_{1,2})\cos(\theta_{2,3})\cos(\theta_{3,1}) - \sum_{\text{cyc}}\cos(\theta_{1,2})\sin(\theta_{2,3})\sin(\theta_{3,1})\\
&= 1+\kappa^3\frac{(-1)^{1+2+1+2+3+1+3+1+1}d_{12}d_{23}d_{31}}{\sqrt{d_{11}d_{22}}\sqrt{d_{22}d_{33}}\sqrt{d_{33}d_{11}}} - \kappa\,\frac{(-1)^{1+2+1}d_{12}}{\sqrt{d_{11}d_{22}}}\frac{\sqrt{\kappa d}}{\sqrt{d_{22}d_{33}}}\frac{\sqrt{\kappa d}}{\sqrt{d_{33}d_{11}}}\\
&\quad - \kappa\,\frac{(-1)^{2+3+1}d_{23}}{\sqrt{d_{22}d_{33}}}\frac{\sqrt{\kappa d}}{\sqrt{d_{33}d_{11}}}\frac{\sqrt{\kappa d}}{\sqrt{d_{11}d_{22}}} - \kappa\,\frac{(-1)^{3+1+1}d_{31}}{\sqrt{d_{33}d_{11}}}\frac{\sqrt{\kappa d}}{\sqrt{d_{11}d_{22}}}\frac{\sqrt{\kappa d}}{\sqrt{d_{22}d_{33}}};
\end{aligned}
$$

note $\sqrt{\kappa d}^2 = \kappa d$ and $\sqrt{d_{11}d_{22}}\sqrt{d_{22}d_{33}}\sqrt{d_{33}d_{11}} = \kappa d_{11}d_{22}d_{33}$;

$$
\begin{aligned}
&= 1-\frac{d_{12}d_{23}d_{31}}{d_{11}d_{22}d_{33}} - \frac{\kappa d(d_{12}+d_{23}-d_{31})}{d_{11}d_{22}d_{33}}\\
&= \frac{d_{11}d_{22}d_{33}-d_{12}d_{23}d_{31}-\kappa d(d_{12}+d_{23}-d_{31})}{d_{11}d_{22}d_{33}}.
\end{aligned}
$$

At this point we draw L for reference:

$$
L = \begin{pmatrix} \kappa & \lambda_{1,2} & \lambda_{3,1}\\ \lambda_{1,2} & \kappa & \lambda_{2,3}\\ \lambda_{3,1} & \lambda_{2,3} & \kappa \end{pmatrix} := \begin{pmatrix} \kappa & \lambda_3 & \lambda_2\\ \lambda_3 & \kappa & \lambda_1\\ \lambda_2 & \lambda_1 & \kappa \end{pmatrix}.
$$

Observe for example that $d_{ii} = 1-\lambda_i^2$, $d_{12} = \kappa\lambda_3 - \lambda_1\lambda_2$, $d_{23} = \kappa\lambda_1 - \lambda_2\lambda_3$, and $d_{31} = \lambda_3\lambda_1 - \kappa\lambda_2$.

Given a_1, a_2, a_3, define the symmetric polynomials $\sigma_1(a) = a_1+a_2+a_3$, $\sigma_2(a) = a_1a_2 + a_2a_3 + a_3a_1$, $\sigma_3(a) = a_1a_2a_3$, and $S_2(a) = a_1^2+a_2^2+a_3^2$. We suppress $\sigma_i(\lambda) = \sigma_i$ and $S_2(\lambda) = S_2$. As an example, we write $\sigma_2(\lambda^2) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$.

Now observe the following facts (recall $\sigma(a^2) = \sigma_2(a)^2 - 2\sigma_1(a)\sigma_3(a)$):

$$
\begin{aligned}
\kappa d &= |\det L|\\
&= 1+2\kappa\sigma_3 - S_2,\\
d_{11}d_{22}d_{33} &= (1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_3^2)\\
&= 1-S_2+\sigma_2(\lambda^2)-\sigma_3^2\\
&= 1-\sigma_1^2+2\sigma_2+\sigma_2^2-2\sigma_1\sigma_3-\sigma_3^2\\
&= (1+\sigma_2)^2-(\sigma_1+\sigma_3)^2,\\
-d_{12}d_{23}d_{31} &= -(\kappa\lambda_3-\lambda_1\lambda_2)(\kappa\lambda_1-\lambda_2\lambda_3)(\lambda_3\lambda_1-\kappa\lambda_2)\\
&= (\kappa\lambda_3-\lambda_1\lambda_2)(\kappa\lambda_1-\lambda_2\lambda_3)(\kappa\lambda_2-\lambda_3\lambda_1)\\
&= \kappa\sigma_3-\sigma_2(\lambda^2)+\kappa\sigma_3S_2-\sigma_3^2,\\
-\kappa d(d_{12}+d_{23}-d_{31}) &= -(1+2\kappa\sigma_3-S_2)(\kappa\sigma_1-\sigma_2),
\end{aligned}
$$

so that

$$
\begin{aligned}
1 - \cos \mathrm{vol}_2 \Delta &= \frac{d_{11}d_{22}d_{33} - d_{12}d_{23}d_{31} - \kappa d(d_{12} + d_{23} - d_{31})}{d_{11}d_{22}d_{33}} \\
&= \frac{1 - S_2 + \sigma_2(\lambda^2) - \sigma_3^2 + \kappa\sigma_3 - \sigma_2(\lambda^2) + \kappa\sigma_3 S_2 - \sigma_3^2 - (1 + 2\kappa\sigma_3 - S_2)(\kappa\sigma_1 - \sigma_2)}{(1+\sigma_2)^2 - (\sigma_1 + \sigma_3)^2} \\
&= \frac{1 - S - 2\sigma_3^2 + \kappa\sigma_3 + \kappa\sigma_3 S_2 - (1 + 2\kappa\sigma_3 - S_2)(\kappa\sigma_1 - \sigma_2)}{(1+\sigma_2)^2 - (\sigma_1 + \sigma_3)^2} \\
&= \frac{(1 + 2\kappa\sigma_3 - S_2)(1 - \kappa\sigma_3) - (1 + 2\kappa\sigma_3 - S_2)(\kappa\sigma_1 - \sigma_2)}{(1+\sigma_2)^2 - (\sigma_1 + \sigma_3)^2} \\
&= \frac{(1 + 2\kappa\sigma_3 - S_2)((1 + \sigma_2) - \kappa(\sigma_1 + \sigma_3))}{(1+\sigma_2)^2 - \kappa^2(\sigma_1 + \sigma_3)^2} \\
&= \frac{1 + 2\kappa\sigma_3 - S_2}{(1 + \sigma_2) + \kappa(\sigma_1 + \sigma_3)} \\
\implies \sin^2\left(\frac{\mathrm{vol}_2 \Delta}{2}\right) &= \frac{1 - \cos \mathrm{vol}_2 \Delta}{2} \\
&= \frac{1 + 2\kappa\sigma_3 - S_2}{2(1 + \kappa\sigma_1 + \sigma_2 + \kappa\sigma_3)} \\
&= \frac{|\det L|}{2(1 + \kappa\lambda_{1,2})(1 + \kappa\lambda_{2,3})(1 + \kappa\lambda_{3,1})},
\end{aligned}
$$

as desired. ■

Although tedious, the above calculations offer a straightforward proof for Tuynman's formula.

4. Developments on Volume Power Series

We give a power series in terms of the length matrix for the volume of an n-simplex convergent in some of both the spherical and the hyperbolic case. First we explain some background; then we review some previous work, which we prove in an alternate fashion and briefly expand upon; then we present our new power series.

The following is well-known in the field.

Cone Integrals in the Spherical Case: Let

$$\omega_{\mathrm{vol}} := \frac{1}{\|x\|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \cancel{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{n+1}$$

be the volume form on a sphere (and also a (half-sheet of a) hyperboloid). Note that this is scale invariant. Kohno and Peiro remark that we have

$$r^n \, \mathrm{d}r \wedge \omega_{\mathrm{vol}} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1},$$

where r is defined as $r := \sqrt{\langle x; x \rangle}$. This can be checked easily by computation. Then we have, for a function f such that $\int_0^\infty r^n f(r^2) \, \mathrm{d}r$ converges, the following relation:

$$\mathrm{vol}_n^+(\Delta^+) \int_0^\infty r^n f(r^2) \, \mathrm{d}r = \int_C f(\langle x; x \rangle) \, \mathrm{d}x = \sqrt{\det L} \int_{\mathbb{R}_+^{n+1}} f(x^\dagger L x) \, \mathrm{d}x, \qquad (*)$$

where C refers to the cone formed by the half-spaces intersecting on the surface of our sphere to form Δ. To see the former equality, note

$$\begin{aligned}
\int_C f(\langle x; x \rangle) \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1} &= \int_C r^n f(\langle x; x \rangle) \, \mathrm{d}r \wedge \omega_{\mathrm{vol}} \\
&= \int_{r=0}^\infty r^n f(r^2) \, \mathrm{d}r \int_{r\Delta} \omega_{\mathrm{vol}} \\
&= \int_{r=0}^\infty r^n f(r^2) \, \mathrm{d}r \int_{\Delta} \omega_{\mathrm{vol}} \\
&= \mathrm{vol}_n^+(\Delta^+) \int_0^\infty r^n f(r^2) \, \mathrm{d}r.
\end{aligned}$$

The latter equality is a change of coordinates $x \longmapsto P^{-1}x$:

$$\begin{aligned}
\int_C \frac{f(\langle x; x \rangle)}{|\det P^{-1}|} |\det P^{-1}| \, \mathrm{d}x &= \int_{P^{-1}(C)} \frac{f\big(\langle Px; Px \rangle\big)}{|\det P^{-1}|} \, \mathrm{d}x \\
&= \int_{\mathbb{R}_+^{n+1}} |\det P| f(x^\dagger P^\dagger P x) \, \mathrm{d}x \\
&= \sqrt{\det L} \int_{\mathbb{R}_+^{n+1}} f(x^\dagger L x) \, \mathrm{d}x,
\end{aligned}$$

where we recall $P^\dagger P = L$ and $\det L > 0$ for spherical simplices. It is clear that the image of C under $x \longmapsto P^{-1}x$ is $\mathbb{R}_+^{n+1}$.

In particular, we can choose $f(x) = e^{-x}$. Recall that

$$\int_0^\infty r^n e^{-r^2} \, \mathrm{d}r = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2},$$

which gives

$$\mathrm{vol}_n^+(\Delta^+) = \frac{2\sqrt{\det L}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger L x} \, \mathrm{d}x. \tag{$*+$}$$

Note that the finiteness of the left-hand side guarantees that the right-hand side converges for all admissible L. (In any case recall that G is positive definite for spherical simplices and therefore so is L, so the exponent is strictly negative.) Recall that in the spherical case L looks like

$$L = \begin{pmatrix} 1 & & \cos l_{ij} \\ & \ddots & \\ \cos l_{ji} & & 1 \end{pmatrix}.$$

Cone Integrals in the Hyperbolic Case: A similar formula holds in the hyperbolic case. The general relation in the hyperbolic case is (again for f such that the left-hand side makes sense):

$$\mathrm{vol}_n^-(\Delta^-) \int_0^\infty r^n f(r^2) \, \mathrm{d}r = \int_C f(-\langle x; x\rangle_-) \, \mathrm{d}x = \sqrt{|\det L|} \int_{\mathbb{R}_+^{n+1}} f(-x^\dagger L x) \, \mathrm{d}x, \tag{$*$}$$

where $\langle \cdot; \cdot \rangle_-$ refers to the Minkowski inner product $\langle x; y\rangle_- \coloneqq x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1} = x^\dagger Q y$, with

$$Q = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

r then refers to $r \coloneqq \sqrt{|\langle x; x\rangle_-|}$. Note that on C, which is contained in the upper light cone, we have $\langle \cdot; \cdot \rangle_- < 0$, so really $r = \sqrt{-\langle x; x\rangle_-}$ as far as we're concerned. To see the former equality, note

$$\begin{aligned}
\int_C f(-\langle x; x\rangle_-) \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1} &= \int_C r^n f(-\langle x; x\rangle) \, \mathrm{d}r \wedge \omega_{\mathrm{vol}} \\
&= \int_{r=0}^\infty r^n f(r^2) \, \mathrm{d}r \int_{r\Delta} \omega_{\mathrm{vol}} \\
&= \int_{r=0}^\infty r^n f(r^2) \, \mathrm{d}r \int_{\Delta} \omega_{\mathrm{vol}} \\
&= \mathrm{vol}_n^-(\Delta^-) \int_0^\infty r^n f(r^2) \, \mathrm{d}r.
\end{aligned}$$

The latter equality also is a change of coordinates $x \longmapsto P^{-1}x$:

$$\begin{aligned}\int_C \frac{f(-\langle x;x\rangle_-)}{|\det P^{-1}|}|\det P^{-1}|\,\mathrm{d}x &= \int_{P^{-1}(C)} \frac{f\big(-\langle Px;Px\rangle_-\big)}{|\det P^{-1}|}\,\mathrm{d}x \\ &= \int_{\mathbb{R}_+^{n+1}} |\det P| f(-x^\dagger P^\dagger Q P x)\,\mathrm{d}x \\ &= \sqrt{|\det L|}\int_{\mathbb{R}_+^{n+1}} f(-x^\dagger L x)\,\mathrm{d}x,\end{aligned}$$

where we recall $P^\dagger Q P = L$ and $\det L < 0$ for hyperbolic simplices. It is clear that the image of C under $x \longmapsto P^{-1}x$ is $\mathbb{R}_+^{n+1}$.

In particular, we can again choose $f(x) = e^{-x}$, which gives

$$\mathrm{vol}_n^-(\Delta^-) = \frac{2\sqrt{|\det L|}}{\Gamma\big(\frac{n+1}{2}\big)}\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger(-L)x}\,\mathrm{d}x. \tag{$*-$}$$

Note the double minus sign. Recall that in the hyperbolic case L looks like

$$-L = \begin{pmatrix} 1 & & \cosh l_{ij} \\ & \ddots & \\ \cosh l_{ji} & & 1 \end{pmatrix}.$$

Since $\cosh \geq 1$, the exponent in the integrand is strictly negative.

Power Series: To summarize, we have

Lemma.

$$\mathrm{vol}_n(\Delta) = \frac{2\sqrt{\kappa \det L}}{\Gamma\big(\frac{n+1}{2}\big)}\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger(\kappa L)x}\,\mathrm{d}x.$$

Hence to get a power series for vol_n it suffices to find a power series for the expression

$$\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T x}\,\mathrm{d}x$$

in terms of the variables t_{ij}, $1 \leq i < j \leq n+1$, where

$$T = \begin{pmatrix} 1 & & t_{ij} \\ & \ddots & \\ t_{ji} & & 1 \end{pmatrix}.$$

Note $t_{ij} = \cos\Big(e^{i\frac{1-\kappa}{4}\pi} l_{ij}\Big)$, where κ is the constant curvature of the space. Perhaps this is better written as $t_{ij} = \cos(\sqrt{\kappa}\, l_{ij})$. First we will find such a power series in a manner which gives an alternate proof for a theorem by Aomoto. In what follows, it should be emphasized that n_{ij} is a sum index, while n is the dimension of the simplex. We will later also use $\vec{n}$ to denote the n_{ij}'s; we stress that $\vec{n}$ and n_{ij} are *not the same thing* as n.

Theorem (Aomoto)**.** For $\Delta^+ \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, the following power series holds around the origin:

$$\begin{aligned}\mathrm{vol}_n^+(\Delta^+) &= \frac{\sqrt{\det L}}{2^n\Gamma\left(\frac{n+1}{2}\right)} \sum_{\substack{n_{ij}=0\\1\le i<j\le n+1}}^{\infty} \cdots \sum \prod_{i=1}^{n+1} \Gamma\left(\frac{1+\sum_{j\neq i} n_{ij}}{2}\right) \prod\prod_{1\le i<j\le n+1} \frac{(-2\lambda_{ij})^{n_{ij}}}{n_{ij}!} \\ &= \frac{\sqrt{\det L}}{2^n\Gamma\left(\frac{n+1}{2}\right)} \sum_{\substack{n_{ij}=0\\1\le i<j\le n+1}}^{\infty} \cdots \sum \prod_{i=1}^{n+1} \Gamma\left(\frac{1+\sum_{j\neq i} n_{ij}}{2}\right) \prod\prod_{1\le i<j\le n+1} \frac{(-2\cos l_{ij})^{n_{ij}}}{n_{ij}!}.\end{aligned}$$

Alternative Proof. This alternate proof will also provide a sufficient condition for convergence. The idea is to write

$$T = \begin{pmatrix} 1 & & t_{ij} \\ & \ddots & \\ t_{ji} & & 1 \end{pmatrix} = \mathrm{Id} + \begin{pmatrix} 0 & & t_{ij} \\ & \ddots & \\ t_{ji} & & 0 \end{pmatrix} = \mathrm{Id} + H,$$

so that

$$\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T x}\,\mathrm{d}x = \int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger x} e^{-x^\dagger H x}\,\mathrm{d}x,$$

write the second factor in the integrand as a power series, then use Fubini's to exchange the order of integration and summation.

Observe that

$$\begin{aligned} e^{-x^\dagger H x} &= e^{-\left(\sum\sum_{1\le i<j\le n+1}(2t_{ij}x_ix_j)\right)} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\sum\sum_{i<j}(-2t_{ij}x_ix_j)\right)^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{|\vec{n}|_1=m} \binom{m}{\vec{n}} \prod\prod_{1\le i<j\le n+1} (-2t_{ij}x_ix_j)^{n_{ij}} \\ &= \sum_{\substack{n_{ij}\ge 0\\1\le i<j\le n+1}} \cdots \sum \left(\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!} \prod_i x_i^{\sum_{j\neq i} n_{ij}} \right) \\ &= \sum_{\vec{n}} \left(\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!} \prod_i x_i^{\sum_{j\neq i} n_{ij}} \right) \end{aligned}$$

($|\vec{n}|_1$ denotes $\sum_{i<j}\sum n_{ij}$), which gives

$$\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T x}\,\mathrm{d}x = \int_{\mathbb{R}_+^{n+1}} \left(\sum_{\vec{n}} e^{-x^\dagger x} \left(\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!} \prod_i x_i^{\sum_{j\neq i} n_{ij}} \right)\right) \mathrm{d}x.$$

We now apply Fubini's theorem. Clearly each $e^{-x^\dagger x}\left(\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j}n_{ij}!}\prod_i x_i^{\sum_{j\neq i}n_{ij}}\right)$ is continuous in $\vec{x}$. Furthermore we have its absolute value is

$$\left|e^{-x^\dagger x}\left(\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j}n_{ij}!}\prod_i x_i^{\sum_{j\neq i}n_{ij}}\right)\right| = e^{-x^\dagger x}\left(\frac{\prod\prod_{i<j}(2|t_{ij}|)^{n_{ij}}}{\prod\prod_{i<j}n_{ij}!}\prod_i x_i^{\sum_{j\neq i}n_{ij}}\right)$$

since $x_i \geq 0$ on $\mathbb{R}_+^{n+1}$, which gives

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}}\left(\sum_{\vec{n}}\left|e^{-x^\dagger x}\left(\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j}n_{ij}!}\prod_i x_i^{\sum_{j\neq i}n_{ij}}\right)\right|\right)\mathrm{d}x \\
&= \int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger x}\exp\left(-x^\dagger\begin{pmatrix} 0 & & -|t_{ij}| \\ & \ddots & \\ -|t_{ji}| & & 0\end{pmatrix}x\right)\mathrm{d}x \\
&= \int_{\mathbb{R}_+^{n+1}} \exp\left(-x^\dagger\begin{pmatrix} 1 & & -|t_{ij}| \\ & \ddots & \\ -|t_{ji}| & & 1\end{pmatrix}x\right)\mathrm{d}x.
\end{aligned}$$

Fubini's condition is that this integral is finite, in which case the order of integration and summation may be switched. It is emphasized here that Fubini's condition essentially replaces a convergence radius calculation: as long as Fubini's condition holds, we are guaranteed that $\int e^{-x^\dagger Tx} = \int\sum f = \sum\int f$ (since the exponential power series $\sum f$ converges uniformly on compact sets), and we know *a priori* that the former must converge, which implies the latter must also converge. Hence the following Fubini condition is sufficient for the power series to converge:

$$\int_{\mathbb{R}_+^{n+1}} \exp\left(-x^\dagger\begin{pmatrix} 1 & & -|t_{ij}| \\ & \ddots & \\ -|t_{ji}| & & 1\end{pmatrix}x\right)\mathrm{d}x < \infty. \tag{$*$}$$

Note that this for example happens when $|t_{ij}| < \frac{1}{n} \quad \forall\, i, j$, as

$$\begin{aligned}
x^\dagger\begin{pmatrix} 1 & & -\frac{1}{n} \\ & \ddots & \\ -\frac{1}{n} & & 1\end{pmatrix}x &= \frac{1}{n}\sum_i nx_i^2 - \frac{1}{n}\sum\sum_{i<j}2x_ix_j \\
&= \frac{1}{n}\sum\sum_{i<j}(x_i - x_j)^2 \\
&\geq 0.
\end{aligned}$$

It should be noted that $t_{ij} = \kappa\lambda_{ij}$, and that $\frac{1}{n} < 1 = \min_{x\in\mathbb{R}}\cosh x$, so it would appear this series might fail to converge in the hyperbolic case. We will speak more of this when we are done computing the power series.

We may then switch the order of integration and summation to obtain

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T x}\,\mathrm{d}x &= \int_{\mathbb{R}_+^{n+1}}\left(\sum_{\vec{n}} e^{-x^\dagger x}\left(\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\neq i} n_{ij}}\right)\right)\mathrm{d}x\\ &= \sum_{\vec{n}}\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\left(\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger x}\prod_i x_i^{\sum_{j\neq i} n_{ij}}\,\mathrm{d}x\right)\\ &= \sum_{\vec{n}}\frac{1}{2^{n+1}}\prod_i \Gamma\left(\frac{1+\sum_{j\neq i} n_{ij}}{2}\right)\frac{\prod\prod_{i<j}(-2t_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!},\end{aligned}$$

where the integral for the coefficient

$$\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger x}\prod_i x_i^{\sum_{j\neq i} n_{ij}}\,\mathrm{d}x = \frac{1}{2^{n+1}}\prod_i \Gamma\left(\frac{1+\sum_{j\neq i} n_{ij}}{2}\right)$$

is a straightforward computation[6]. For $\kappa = 1$ this returns (recall $\mathrm{vol}_n^+\,\Delta^+ = \frac{2\sqrt{\det L}}{\Gamma\left(\frac{n+1}{2}\right)}\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger L x}\,\mathrm{d}x$)

$$\mathrm{vol}_n^+\,\Delta^+ = \frac{\sqrt{\det L}}{2^n\Gamma\left(\frac{n+1}{2}\right)}\sum_{\vec{n}}\prod_i \Gamma\left(\frac{1+\sum_{j\neq i} n_{ij}}{2}\right)\frac{\prod\prod_{i<j}(-2\lambda_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!},$$

as Aomoto claimed[7]. In the hyperbolic case, we have $|t_{ij}| \geq 1$; however, even for the boundary case $t_{ij} = \kappa\lambda_{ij} = \cosh l_{ij} = 1 \quad \forall\, i, j$, the terms of this series fail to go to zero as $|\vec{n}|_1 \to \infty$ (straightforward Stirling calculation), implying divergence. Hence this series only holds in a subset of the spherical case. ■

Result: It is a simple modification of the above idea to write

$$T = \begin{pmatrix} 1 & & 1\\ & \ddots & \\ 1 & & 1\end{pmatrix} + \begin{pmatrix} 0 & & t_{ij}-1\\ & \ddots & \\ t_{ji}-1 & & 0\end{pmatrix} = T_0 + H.$$

Let the entries of H be denoted $\eta_{ij} = t_{ij} - 1$ for sake of brevity[8].

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[6] I should mention the existence of a *method of brackets*, a “generalization” of Ramanujan's Master Theorem which is not yet entirely on rigorous footing. While not used in this report, this methodology is very interesting.

[7] There were two minor typos in Aomoto's paper: that the Gamma function was missing a +1 from the numerator of its argument, and that the $\det L$ was missing a square root. These are corrected here.

[8] η stands for error

Observe that

$$\begin{aligned}
e^{-x^\dagger H x} &= e^{-\left(\sum\sum_{1\le i<j\le n+1} (2\eta_{ij}x_ix_j)\right)} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\sum\sum_{i<j}(-2\eta_{ij}x_ix_j)\right)^m \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{|\vec{n}|_1=m} \binom{m}{\vec{n}} \prod\prod_{1\le i<j\le n+1} (-2\eta_{ij}x_ix_j)^{n_{ij}} \\
&= \sum_{\substack{n_{ij}\ge 0 \\ 1\le i<j\le n+1}}\cdots\sum \left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\ne i} n_{ij}}\right) \\
&= \sum_{\vec{n}} \left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\ne i} n_{ij}}\right),
\end{aligned}$$

which gives

$$\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T x}\,\mathrm{d}x = \int_{\mathbb{R}_+^{n+1}} \left(\sum_{\vec{n}} e^{-x^\dagger T_0 x}\left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\ne i} n_{ij}}\right)\right)\mathrm{d}x.$$

We now apply Fubini's theorem. Clearly each $e^{-x^\dagger T_0 x}\left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\ne i} n_{ij}}\right)$ is continuous in $\vec{x}$. Furthermore we have its absolute value is

$$\left|e^{-x^\dagger T_0 x}\left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\ne i} n_{ij}}\right)\right| = e^{-x^\dagger T_0 x}\left(\frac{\prod\prod_{i<j}(2|\eta_{ij}|)^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\ne i} n_{ij}}\right)$$

since $x_i \ge 0$ on $\mathbb{R}_+^{n+1}$, which gives

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \left(\sum_{\vec{n}}\left|e^{-x^\dagger T_0 x}\left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!}\prod_i x_i^{\sum_{j\ne i} n_{ij}}\right)\right|\right)\mathrm{d}x \\
&= \int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T_0 x}\exp\left(x^\dagger \begin{pmatrix} 0 & & |\eta_{ij}| \\ & \ddots & \\ |\eta_{ji}| & & 0 \end{pmatrix} x\right)\mathrm{d}x \\
&= \int_{\mathbb{R}_+^{n+1}} \exp\left(-x^\dagger \begin{pmatrix} 1 & & 1-|t_{ij}-1| \\ & \ddots & \\ 1-|t_{ji}-1| & & 1 \end{pmatrix} x\right)\mathrm{d}x,
\end{aligned}$$

so that this time Fubini's condition is

$$\int_{\mathbb{R}_+^{n+1}} \exp\left(-x^\dagger \begin{pmatrix} 1 & & 1-|t_{ij}-1| \\ & \ddots & \\ 1-|t_{ji}-1| & & 1 \end{pmatrix} x\right)\mathrm{d}x < \infty. \qquad (*)$$

Note again that if $1 - |t_{ij} - 1| > -\frac{1}{n} \iff |t_{ij} - 1| < 1 + \frac{1}{n} \iff -\frac{1}{n} < t_{ij} < 2 + \frac{1}{n}$, then the integral above converges. Hence if $|t_{ij} - 1| < 1 + \frac{1}{n}$ then Fubini's theorem applies and we can switch the order of the sum and the integral to obtain

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T x}\,\mathrm{d}x &= \sum_{\vec{n}} \left(\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T_0 x} \left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!} \prod_i x_i^{\sum_{j\neq i} n_{ij}} \right) \mathrm{d}x \right) \\ &= \sum_{\vec{n}} \left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!} \int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T_0 x} \prod_i x_i^{\sum_{j\neq i} n_{ij}}\,\mathrm{d}x \right).\end{aligned}$$

It is again emphasized that as long as Fubini's criterion holds (i.e. $\int \sum |f| < \infty$), which is the case when for example $|t_{ij} - 1| < 1 + \frac{1}{n}$, it must be the case that the power series on the right hand side converges. This side-steps the need for a computation for the radius of convergence later. It remains to compute the coefficients $\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T_0 x} \prod_i x_i^{\sum_{j\neq i} n_{ij}}\,\mathrm{d}x$.

Denote $c_i \coloneqq \sum_{j\neq i} n_{ij}$ and observe that $x^\dagger T_0 x = \sum\limits_i x_i^2 + \sum\limits_{i<j}\sum 2x_i x_j = (x_1 + \cdots + x_{n+1})^2$. Let us change variables $x_1 + \cdots + x_{n+1} = \sigma$, which gives

$$\begin{aligned}&\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T_0 x} \prod_i x_i^{\sum_{j\neq i} n_{ij}}\,\mathrm{d}x \\ &= \int_0^\infty \cdots \int_0^\infty x_1^{c_1} \cdots x_{n+1}^{c_{n+1}} e^{-(x_1+\cdots+x_{n+1})^2}\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n+1} \\ &= \int_0^\infty \int_0^\sigma \int_0^{\sigma - x_n} \cdots \int_0^{\sigma - x_n - \cdots - x_2} x_1^{c_1} \cdots x_n^{c_n} (\sigma - x_n - \cdots - x_2 - x_1)^{c_{n+1}} e^{-\sigma^2}\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}\,\mathrm{d}x_n\,\mathrm{d}\sigma.\end{aligned}$$

This computation reduces to calculating

$$\int_0^s x^a (s-x)^b\,\mathrm{d}x = \frac{a!b!}{(a+b+1)!} s^{a+b+1},$$

which can be verified via WolframAlpha, and iterating integrals of this form. Doing so affords the following:

$$\begin{aligned}&= \int_0^\infty \frac{c_1! \cdots c_{n+1}!}{(c_1 + \cdots + c_{n+1} + n)!} \sigma^{c_1 + \cdots + c_{n+1} + n} e^{-\sigma^2}\,\mathrm{d}\sigma \\ &= \frac{1}{2} \frac{c_1! \cdots c_{n+1}!}{(c_1 + \cdots + c_{n+1} + n)!} \Gamma\left(\frac{n + 1 + c_1 + \cdots + c_{n+1}}{2} \right) \\ &= \frac{1}{2} \frac{\prod_i \Gamma\left(1 + \sum_{j\neq i} n_{ij}\right)}{\Gamma(n + 1 + 2|\vec{n}|_1)} \Gamma\left(\frac{n+1}{2} + |\vec{n}|_1 \right).\end{aligned}$$

With these coefficients we have

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T x}\,\mathrm{d}x &= \sum_{\vec{n}} \left(\frac{\prod\prod_{i<j}(-2\eta_{ij})^{n_{ij}}}{\prod\prod_{i<j} n_{ij}!} \cdot \frac{1}{2} \frac{\prod_i \Gamma\left(1 + \sum_{j\neq i} n_{ij}\right)}{\Gamma(n + 1 + 2|\vec{n}|_1)} \Gamma\left(\frac{n+1}{2} + |\vec{n}|_1 \right) \right) \\ &= \sum_{\vec{n}} \left(\frac{1}{2} (-2)^{|\vec{n}|_1} \frac{\Gamma\left(\frac{n+1}{2} + |\vec{n}|_1\right)}{\Gamma(n + 1 + 2|\vec{n}|_1)} \prod_i \Gamma\left(1 + \sum_{j\neq i} n_{ij} \right) \right) \frac{\vec{\eta}^{\vec{n}}}{\vec{n}!}. \qquad (*)\end{aligned}$$

Plugging this into our formula for vol_n from earlier, we get

Theorem.

$$\mathrm{vol}_n(\Delta) = \frac{\sqrt{\kappa \det L}}{\Gamma\left(\frac{n+1}{2}\right)} \sum_{\vec{n}} \left(\frac{(-2)^{|\vec{n}|_1} \Gamma\left(\frac{n+1}{2} + |\vec{n}|_1\right)}{\Gamma(n+1+2|\vec{n}|_1)} \prod_{i=1}^{n+1} \Gamma\left(1 + \sum_{j \neq i} n_{ij}\right)\right) \frac{(\kappa\vec{\lambda} - \vec{1})^{\vec{n}}}{\vec{n}!},$$

where κ is the constant curvature of the space, $\lambda_{ij} = \kappa \cos(\sqrt{\kappa}\, l_{ij})$ is the ij-th entry of the length matrix L, and $\vec{1}$ is the vector of all ones. This power series converges if

$$\int_{\mathbb{R}_+^{n+1}} \exp\left(-x^\dagger \begin{pmatrix} 1 & & 1 - |\kappa\lambda_{ij} - 1| \\ & \ddots & \\ 1 - |\kappa\lambda_{ji} - 1| & & 1 \end{pmatrix} x\right) \mathrm{d}x < \infty,$$

which happens for example if every

$$|\kappa\lambda_{ij} - 1| < 1 + \frac{1}{n} \iff -\frac{1}{n} < \kappa\lambda_{ij} = \cos\left(\sqrt{\kappa}\, l_{ij}\right) < 2 + \frac{1}{n} \qquad \forall\, i, j.$$

This above derivation hinged upon the very special fact about the base matrix T_0, which is that $x^\dagger T_0 x = \sigma_1^2$. This yields a very simple integrand (which can be dealt with by, as we saw, a linear change of variables) for computing the power series coefficients. For a different base point, say

$$T_0 = \begin{pmatrix} 1 & & 2 \\ & \ddots & \\ 2 & & 1 \end{pmatrix},$$

this nice reduction does not apply, and calculating $\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger T_0 x} \prod_i x_i^{\sum_{j \neq i} n_{ij}}\, \mathrm{d}x$ is not nearly as easy.

5. Ideal Hyperbolic Simplices

We present some work on volumes of ideal hyperbolic simplices and discuss future direction. The work in this section will be in many ways reminiscent of that of the last section.

Previous Work: Given a cone C in $\mathbb{R}^{n+1}$, we can consider its intersection with both $\mathbb{H}^n$ and $\mathbb{E}^n$ (the plane $x_{n+1} = 1$). The former is a hyperbolic simplex while the latter is a Euclidean simplex. Recall our construction of C given the $\{p_i\}$.

If the simplex Δ^- is moreover ideal, that is if all its vertices lie on $\partial_\infty \mathbb{H}^n$ the boundary at infinity of $\mathbb{H}^n$, then the vectors determining C lie on the light cone, which is the set of points in $\mathbb{R}^{n,1}$ such that $\langle x; x\rangle_- = 0$. In the ideal case we denote C_∞ to be the cone of an ideal simplex Δ^-_∞.

A simplex is regular if, for any permutation of its vertices, there exists an isometry of the space bringing the simplex to that permutation of its vertices. It is a fact that ideal regular simplices are unique up to isometry; hence it makes sense to speak of *the* $\Delta^-_{\infty,\mathrm{reg}}$ and $C_{\infty,\mathrm{reg}}$. Haagerup and Munkholm showed that, in the regular case,

Theorem (Haagerup and Munkholm)**.** Let $C_{\infty,\mathrm{reg}}$ denote the cone whose intersection with $\mathbb{H}^n$ gives the ideal regular hyperbolic simplex (unique up to isometry). Then

$$\left(\frac{n+2}{n+1}\right)^{\frac{n-1}{2}} \leq \frac{n-1}{n}\frac{\mathrm{vol}^-_n(C_{\infty,\mathrm{reg}} \cap \mathbb{H}^n)}{\mathrm{vol}^0_n(C_{\infty,\mathrm{reg}} \cap \mathbb{E}^n)} \leq \left(\frac{n}{n-1}\right)^{\frac{n-1}{2}},$$

which implies

$$\lim_{n\to\infty} \frac{\mathrm{vol}^-_n(C_{\infty,\mathrm{reg}} \cap \mathbb{H}^n)}{\mathrm{vol}^0_n(C_{\infty,\mathrm{reg}} \cap \mathbb{E}^n)} = \sqrt{e}.$$

We investigate what happens when this regularity condition is loosened.

Cone Integrals: An ideal cone C_∞ can be characterized by $n+1$ vectors lying on the light cone, $v_1, \cdots, v_{n+1}$, so $\langle v_i; v_i\rangle_- = 0$. Without loss of generality, let each such vector be of form

$$v_i = \begin{pmatrix} v_{i,1} \\ \vdots \\ v_{i,n} \\ 1 \end{pmatrix},$$

where $v_{i,1}^2 + \cdots + v_{i,n}^2 = 1$. Let

$$V \coloneqq \begin{pmatrix} \uparrow & \cdots & \uparrow \\ v_1 & \cdots & v_{n+1} \\ \downarrow & \cdots & \downarrow \end{pmatrix}.$$

Recall

$$\mathrm{vol}^-_n(\Delta^-_\infty)\int_0^\infty r^n f(r^2)\,\mathrm{d}r = \int_{C_\infty} f(-\langle x; x\rangle_-)\,\mathrm{d}x = |\det V| \int_{\mathbb{R}^{n+1}_+} f(-x^\dagger V^\dagger Q V x)\,\mathrm{d}x$$

from last section (all the proofs work *verbatim* when we replace P with V). Picking $f(x) = e^{-x}$ and noting that, for $\cos\phi_{ij} := v_{i,1}v_{j,1} + \cdots + v_{i,n}v_{j,n}$,

$$-\Phi := V^\dagger Q V = \begin{pmatrix} 0 & & \cos\phi_{ij} - 1 \\ & \ddots & \\ \cos\phi_{ij} - 1 & & 0 \end{pmatrix},$$

we get

$$\begin{aligned} \mathrm{vol}_n^-(\Delta_\infty^-) &= \frac{2|\det V|}{\Gamma\left(\frac{n+1}{2}\right)} \int_{\mathbb{R}_+^{n+1}} \exp\left(-x^\dagger \begin{pmatrix} 0 & & 1 - \cos\phi_{ij} \\ & \ddots & \\ 1 - \cos\phi_{ij} & & 0 \end{pmatrix} x \right) \mathrm{d}x \\ &= \frac{2|\det V|}{\Gamma\left(\frac{n+1}{2}\right)} \int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger \Phi x} \, \mathrm{d}x. \end{aligned}$$

Observe that the entries of Φ are nonnegative.

Volume Ratios: Denote $\Delta_\infty^0 = C_\infty \cap \mathbb{E}^n$ the Klein Euclidean projection of Δ_∞^-. Observe the vertices of Δ_∞^0 are precisely given by $\{v_i\}$, and there is already a row of ones in V. Recall then that the Euclidean volume of a Euclidean simplex is given by

$$\mathrm{vol}_n^0(\Delta^0) = \frac{|\det V|}{n!},$$

so that

$$\frac{\mathrm{vol}_n^-(\Delta_\infty^-)}{\mathrm{vol}_n^0(\Delta_\infty^0)} = \frac{2 \cdot n!}{\Gamma\left(\frac{n+1}{2}\right)} \int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger \Phi x} \, \mathrm{d}x. \tag{$*$}$$

Let Δ_{std} be the Euclidean n-simplex in $\mathbb{R}^{n+1}$ formed by vertices $e_1, \cdots, e_{n+1}$ and $\omega_{\Delta_{\mathrm{std}}}$ be the scale-invariant volume form restricted to Δ_{std}. Then $\mathrm{vol}_n^0(\Delta_{\mathrm{std}}) = \frac{\sqrt{n+1}}{n!}$ and the volume form is explicitly given by $\omega_{\Delta_{\mathrm{std}}} = \frac{*\mathrm{d}x_1 + \cdots + *\mathrm{d}x_{n+1}}{\sigma_1^n \sqrt{n+1}}$, where as usual $\sigma_1 = x_1 + \cdots + x_{n+1}$. We found that

Lemma. The ratio

$$\frac{\mathrm{vol}_n^-(\Delta_\infty^-)}{\mathrm{vol}_n^0(\Delta_\infty^0)} = \frac{1}{\mathrm{vol}_n^0(\Delta_{\mathrm{std}})} \int_{\Delta_{\mathrm{std}}} (x^\dagger \Phi x)^{-\frac{n+1}{2}} \, \omega_{\Delta_{\mathrm{std}}}$$

is the average of $(x^\dagger \Phi x)^{-\frac{n+1}{2}}$ over Δ_{std}, the standard Euclidean n-simplex in $\mathbb{R}^{n+1}$.

Proof. That

$$\omega_{\Delta_{\mathrm{std}}} = \frac{*\mathrm{d}x_1 + \cdots + *\mathrm{d}x_{n+1}}{\sigma_1^n \sqrt{n+1}}$$

and

$$\mathrm{vol}_n^0(\Delta_{\mathrm{std}}) = \frac{\sqrt{n+1}}{n!}$$

are simple checks. Note that

$$\frac{\sigma_1^n}{\sqrt{n+1}} \mathrm{d}\sigma_1 \wedge \omega_{\Delta_{\mathrm{std}}} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1} = \mathrm{d}x.$$

Then

$$\int_{\mathbb{R}_+^{n+1}} e^{-x^\dagger \Phi x}\,\mathrm{d}x = \int_{x\in\Delta_{\mathrm{std}}}\int_{\sigma_1=0}^{\infty} \frac{\sigma_1^n}{\sqrt{n+1}} e^{-\sigma_1^2 x^\dagger \Phi x}\,\mathrm{d}\sigma_1 \wedge \omega_{\Delta_{\mathrm{std}}}$$
$$= \frac{1}{\sqrt{n+1}} \int_{x\in\Delta_{\mathrm{std}}} \left(\int_{\sigma_1=0}^{\infty} \sigma_1^n e^{-(x^\dagger\Phi x)\sigma_1^2}\,\mathrm{d}\sigma_1\right)\omega_{\Delta_{\mathrm{std}}}$$
$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\sqrt{n+1}} \int_{x\in\Delta_{\mathrm{std}}} (x^\dagger \Phi x)^{-\frac{n+1}{2}}\,\omega_{\Delta_{\mathrm{std}}},$$

so that

$$\frac{\mathrm{vol}_n^-(\Delta_\infty^-)}{\mathrm{vol}_n^0(\Delta_\infty^0)} = \frac{2\cdot n!}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\sqrt{n+1}} \int_{x\in\Delta_{\mathrm{std}}} (x^\dagger\Phi x)^{-\frac{n+1}{2}}\,\omega_{\Delta_{\mathrm{std}}} = \frac{1}{\mathrm{vol}_n^0(\Delta_{\mathrm{std}})} \int_{\Delta_{\mathrm{std}}} (x^\dagger\Phi x)^{-\frac{n+1}{2}}\,\omega_{\Delta_{\mathrm{std}}},$$

as claimed. ■

It should be noted that, in the general hyperbolic simplex case (not necessarily ideal or regular), the same procedure as above gives

Lemma. The ratio

$$\frac{\mathrm{vol}_n^-(\Delta^-)}{\mathrm{vol}_n^0(\Delta^0)} = \frac{1}{\mathrm{vol}_n^0(\Delta_{\mathrm{std}})} \int_{\Delta_{\mathrm{std}}} (-x^\dagger L x)^{-\frac{n+1}{2}}\,\omega_{\Delta_{\mathrm{std}}}$$

is the average of $(-x^\dagger L x)^{-\frac{n+1}{2}}$ over Δ_{std}, the standard Euclidean n-simplex in $\mathbb{R}^{n+1}$.

Recall $-L$ has strictly positive entries in the hyperbolic case. Incidentally the above gives yet another alternative proof for our result in the last section by taking $-x^\dagger L x = x^\dagger T_0 x + x^\dagger H x = \sigma_1^2 + x^\dagger H x = 1 + x^\dagger H x$ and using Newton's binomial expansion.

Future Direction: The hope now is to do something similar to the alternative proof outlined above in the ideal case. We can write

$$x^\dagger \Phi x = x^\dagger \begin{pmatrix} 1 & & 1 \\ & \ddots & \\ 1 & & 1 \end{pmatrix} x - x^\dagger \begin{pmatrix} 1 & & \cos\phi_{ij} \\ & \ddots & \\ \cos\phi_{ji} & & 1 \end{pmatrix} x = \sigma_1^2 - x^\dagger G_\phi x = 1 - x^\dagger G_\phi x,$$

so that

$$\int_{\Delta_{\mathrm{std}}} (x^\dagger\Phi x)^{-\frac{n+1}{2}}\,\omega_{\Delta_{\mathrm{std}}} = \int_{\Delta_{\mathrm{std}}} (1 - x^\dagger G_\phi x)^{-\frac{n+1}{2}}\,\omega_{\Delta_{\mathrm{std}}};$$

then one can conceivably use Newton's binomial theorem and expand this as a power series in $\cos\phi_{ij}$ (centered either around 0 or around $\frac{1}{n}$, which is the regular case). Hopefully bounding this power series will yield results on bounds for the ratio $\frac{\mathrm{vol}_n^-(\Delta_\infty^-)}{\mathrm{vol}_n^0(\Delta_\infty^0)}$.

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