Conformal dimension and energy of graph maps

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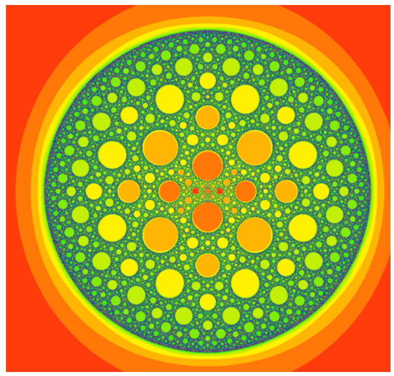
Let J be the set of points in the complex plane which do not converge to infinity under iteration of the rational function $f(z) = \frac{4}{27}\frac{(z^2-z+1)^3}{(z(z-1))^2}$, considered as a map from the complex plane $\mathbb{C}$ to itself. The set J is a fractal; it looks like the dark part in the figure below:



Suppose $1 \leq p \leq 2$. The p-dimensional Hausdorff measure $\mathcal{H}^p$ generalizes the notion of length ($p = 1$) and area ($p = 2$): we have $\mathcal{H}^p(B(x,r)) \asymp r^p$ for all balls $B(x,p)$. The function $p \mapsto \mathcal{H}^p(J)$ is a little crazy: there's a magic number, the *Hausdorff dimension* $\operatorname{hdim}(J)$ of J, where the Hausdorff measure switches from being 0 to being ∞:

$$\sup\{p : \mathcal{H}^p(J) = \infty\} = \operatorname{hdim}(J) = \inf\{p : \mathcal{H}^p(J) = 0\}.$$

The Hausdorff dimension measures "how fractal" J is, and is between 1 (e.g., a smooth circle) and 2 (e.g., the whole plane).

It's known that if we perturb the coefficients of f a little bit, then there is a corresponding fractal, and the Hausdorff dimension is continuous at f. So we could ask how small we could make the Hausdorff dimension while maintaining the general structure of J. We could also consider more exotic types of perturbations where we keep the formula but change the metric and measure. We imagine replacing the Euclidean metric with some other metric. We don't want to be too crazy: we'd like Euclidean balls to be almost round in the new metric. We replace the Euclidean Hausdorff measures $\mathcal{H}^p$ with other related measures μ^p for which we still have $\mu^p((B(x,r))) \asymp r^p$. How small can we make p?

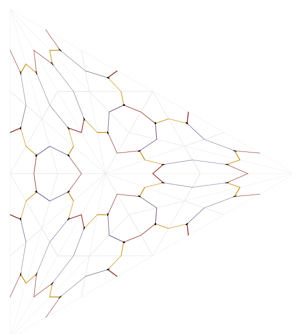
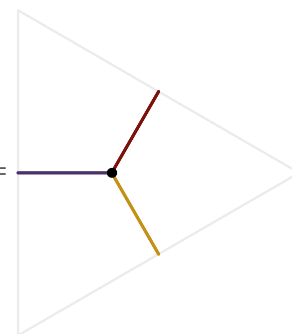
The smallest such p is called the *conformal dimension* $\operatorname{confdim}(J)$ of J. It is known that

$$1 < \operatorname{confdim}(J) \leq \operatorname{hdim}(J) < 2.$$

There's a lot of interest in calculating and estimating these numbers. We've developed an approach to this, and would like to see how it might work with your help.

Last summer, students Baptiste deJean and Christian Gorski studied the following problem. Suppose you have two finite graphs G_1 and G_2 in which the edges have finite lengths. Fix your favorite family $\mathcal{F}$ of piecewise-linear (read: "nice") maps $\phi : G_1 \to G_2$. There's a kind of energy we can define: pick a point y in the target G_2, look at the $x_1, x_2, \ldots$ that map to y under ϕ (we can assume this is a finite set), look at $|f'(x_1)| + |f'(x_2)| + \ldots$, and find the biggest this can possibly be as y varies: this is called $E_2^2(\phi)$. Now try to minimize $E_2^2(\phi)$ for as ϕ varies over $\mathcal{F}$. There's a whole story about how this connects to rubber bands and resistor networks and you can read about it in their report here: `http://www.math.indiana.edu/reu/2016/reu2016.pdf`. This involves *linear convex optimization*, about which a lot is known.

For $1 \leq p \leq 2$ let's now consider a different energy, E_p^p, where in the definition we use the formula $|f'(x_1)|^{p-1} + |f'(x_2)|^{p-1} + \ldots$ instead of $|f'(x_1)| + |f'(x_2)| + \ldots$. Again we try to minimize this new energy $E_p^p(\phi)$ as ϕ varies within our family $\mathcal{F}$. We think this will help us estimate $\operatorname{confdim}(J)$ if we take

$G_1 =$  $G_2 =$ 

and take $\mathcal{F}$ to be the family of maps which sort of squish G_1 onto G_2 in the obvious way if you overlay the two figures, mapping, for instance, all the vertices on the left edge of G_1 onto the single vertex on the left edge of G_2. If $p \neq 2$ then minimizing E_p^p now involves *nonlinear convex optimization* and this is harder.

The above example might be a little too complicated to start with, so the project will likely start with simpler examples such as the ones considered by deJean and Gorski.

Prerequisites: Multivariable calculus and linear algebra are essential. Programming experience will be very helpful.