VERTICAL AVERAGED VELOCITY IN BÉNARD CONVECTION

An REU project
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1. In a nutshell

This project is suitable for a student with strong computational skills and a keen interest in partial differential equations from fluid dynamics. The question is whether the vertical average of a 3D fluid driven by a temperature imbalance on the boundary displays the features of 2D turbulence. Toward an answer for this, we would carry out direct numerical simulation of the 3D problem and then extract from that solution certain quantities which determine the nature of the body force in the equation for the vertically averaged velocity.

The simulation of the 3D problem can be done by Dedalus

http://dedalus-project.org/about.html

a suite of Python scripts which call computational modules. In fact there is an example specifically for the problem of interest. The task then, is to construct the body force for the vertically averaged velocity. Since the relevant quantities are defined in terms of derivatives and integrals (see (2.8)-(2.10) below) and Dedalus is a spectral code, this should be straightforward. The student would then interpret the results in light of our previous work which explains the criteria for turbulence. We have already done analysis to determine an upper bound on this body force. Without a meaningful lower bound however, we cannot say if the force is strong enough to support 2D turbulence. This is why we turn to simulations.

What follows are technical details, which are not necessarily to be fully understood at this time, but should give a flavor of the project. The student need not have prior exposure to the theories of turbulence. Good computational skills, some exposure to PDEs and an open mind are essential.

2. What to compute

We want to solve the 3D Rayleigh-Benard problem

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = -g\mathbf{k}(1-\alpha\delta_T(1+T)), \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T - \beta\Delta T = 0, \tag{2.3}$$

supplemented with the boundary conditions

$$\mathbf{u} = 0 \text{ at } z = 0 \text{ and } z = h, \tag{2.4}$$

$$T(0) = 1, \quad T(h) = 0, \tag{2.5}$$

$$p, \mathbf{u}, T \text{ periodic in horizontal directions } x, y, \tag{2.6}$$

in the domain

$$\Omega = \{(x,y,z) : (x,y) \in [0,L] \times [0,L], 0 \le z \le h\}. \tag{2.7}$$

We write $\mathbf{u} = (u, v, w)$.

The unusual thing is that in the process of solving (2.1)-(2.3), we also want to compute the quantities

$$\frac{\partial u}{\partial z}(x,y,h)\ , \quad \frac{\partial u}{\partial z}(x,y,0)\ , \quad \frac{\partial v}{\partial z}(x,y,h)\ , \quad \frac{\partial v}{\partial z}(x,y,0) \tag{2.8}$$

as well as

$$\int_0^h \frac{\partial}{\partial x}\left[(u-\tilde{u})^2\right] dz\ , \quad \int_0^h \frac{\partial}{\partial y}[(v-\tilde{v})(u-\tilde{u})]dz \tag{2.9}$$

and

$$\int_0^h \frac{\partial}{\partial x}[(u-\tilde{u})(v-\tilde{v})]dz\ , \quad \int_0^h \frac{\partial}{\partial y}\left[(v-\tilde{v})^2\right] dz\ . \tag{2.10}$$

where

$$\tilde{u} = \tilde{u}(x,y) = \frac{1}{h}\int_0^h u(x,y,z)\ dz\ ,$$

and similarly for v.

3. Why

What we are after is the Grashof number

$$G = \sup_t \frac{L^2\|\mathbf{F}_2\|}{\nu^2 4\pi^2}$$

where $\mathbf{F}_2$ is force for the 2D Navier-Stokes system that results when the 3D Rayleigh-Benard system is vertically averaged. The force $\mathbf{F}_2$ is defined in (3.8). Here $\|\cdot\|$ is the L^2-norm in the horizontal spatial variables. This boils down to computing the norms of the terms in (3.4), (3.6) and (3.7) while solving (2.1).

Taking the vertical spatial average of the three dimensional system of momentum equations, we obtain

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} - \nu\Delta\tilde{u} + \frac{\partial \tilde{p}}{\partial x} = F_u^{\mathrm{R}} + F_u^{\partial} \tag{3.1}$$

$$\frac{\partial \tilde{v}}{\partial t} + \tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y} - \nu\Delta\tilde{v} + \frac{\partial \tilde{p}}{\partial y} = F_v^{\mathrm{R}} + F_v^{\partial}\ , \tag{3.2}$$

where

$$\tilde{u} = \tilde{u}(x,y) = \frac{1}{h}\int_0^h u(x,y,z)\ dz\ ,$$

and similarly for v. Due to (2.2) and (2.4) this averaged flow is divergence free:

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0. \tag{3.3}$$

We identify the shear force due to boundary terms as

$$\begin{aligned} \mathbf{F}^{\partial} &= (F_u^{\partial}, F_v^{\partial}) \\ &= \frac{\nu}{h}\left(\frac{\partial u}{\partial z}(x,y,h) - \frac{\partial u}{\partial z}(x,y,0), \frac{\partial v}{\partial z}(x,y,h) - \frac{\partial v}{\partial z}(x,y,0)\right)\ . \end{aligned} \tag{3.4}$$

In order to express the Reynolds stress related force, $\mathbf{F}^{\mathrm{R}} = (F_u^{\mathrm{R}}, F_v^{\mathrm{R}})$, we first integrate by parts and apply (2.4) and (2.2) to obtain

$$\int_0^h w\frac{\partial u}{\partial z}dz = -\int_0^h u\frac{\partial w}{\partial z}dz = \int_0^h u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz\ . \tag{3.5}$$

Using (3.5) and (3.3), we write

$$\begin{aligned}
-F_u^{\mathrm{R}} &= \frac{1}{h}\int_0^h u\frac{\partial u}{\partial x}dz + \frac{1}{h}\int_0^h v\frac{\partial u}{\partial y}dz + \frac{1}{h}\int_0^h w\frac{\partial u}{\partial z}dz - \tilde{u}\frac{\partial \tilde{u}}{\partial x} - \tilde{v}\frac{\partial \tilde{u}}{\partial y} \\
&= \frac{2}{h}\int_0^h u\frac{\partial u}{\partial x}dz + \frac{1}{h}\int_0^h \left(u\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial y}\right)dz - 2\tilde{u}\frac{\partial \tilde{u}}{\partial x} - \tilde{v}\frac{\partial \tilde{u}}{\partial y} - \tilde{u}\frac{\partial \tilde{v}}{\partial y} \\
&= \frac{\partial}{\partial x}(\widetilde{u^2}-\tilde{u}^2) + \frac{\partial}{\partial y}(\widetilde{uv}-\tilde{u}\tilde{v}) \\
&= \frac{1}{h}\int_0^h \frac{\partial}{\partial x}\left[(u-\tilde{u})^2\right]dz + \frac{1}{h}\int_0^h \frac{\partial}{\partial y}[(v-\tilde{v})(u-\tilde{u})]dz\ .
\end{aligned} \tag{3.6}$$

Similarly we find that

$$-F_v^{\mathrm{R}} = \frac{1}{h}\int_0^h \frac{\partial}{\partial x}[(u-\tilde{u})(v-\tilde{v})]dz + \frac{1}{h}\int_0^h \frac{\partial}{\partial y}\left[(v-\tilde{v})^2\right]dz\ . \tag{3.7}$$

In periodic problems it is convenient to work with solutions that have zero mean. From its final form it is easy to see that the stress related force has zero spatial average. Since, however, this need not be the case for the boundary force we split the vertical average of the velocity as $\tilde{\mathbf{u}} = \mathbf{u}_2 + \mathbf{u}_0$ where

$$\mathbf{u}_0 = \frac{1}{L^2}\iint_{[0,L]^2} \tilde{\mathbf{u}}\,dx\,dy = \frac{1}{hL^2}\iiint_{[0,L]^2\times[0,h]} (u,v)\,dx\,dy\,dz,$$

and obtain

$$\frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2\cdot\nabla_2)\mathbf{u}_2 - \nu\Delta_2\mathbf{u}_2 + \nabla_2\tilde{p} = \mathbf{F}_2$$

where

$$\mathbf{F}_2 = (\mathbf{u}_0\cdot\nabla_2)\tilde{\mathbf{u}} + \mathbf{F}^{\mathrm{R}} + \frac{1}{L^2}\iint_{[0,L]^2} \mathbf{F}^{\partial}\ dx\ dy\ . \tag{3.8}$$

Instead of working with $\tilde{\mathbf{u}}$, we work with $\mathbf{u}_2$, which has zero space average.