# AN ALGORITHM FOR POLARIZING TYPED COMBINATORS

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Abstract. Previous work has examined the use of polarity marked CCG combinators to natural logic inferences. To contribute to this work, we present a proof system to add polarity markings to proper typed combinators. The proof system takes a proper typed combinator in the form of an application tree and uses the polarizing rules of application and withdrawal to polarize the combinator. We prove that the polarizing rule of application is sound and conjecture that the polarizing rule of withdrawal is sound as well, inductively resulting in a polarized rule for the typed combinator that we conjecture is sound as well. We also present an algorithm for applying this proof system to a subset of proper typed combinators. [1]

## 1. Introduction

The study of natural logic seeks to make logical inferences from natural language. One approach, monotonicity reasoning, makes logical inferences from natural language by replacing constituents with more or less general constituents [1]. For example, if we know that 'not every conference-goer$^{\downarrow}$ fidgets$^{\uparrow}$', we can replace the word 'conference-goer' with the less general phrase 'bored conference-goer' and replace the word 'fidgets' with the more general word 'moves' to infer that 'not every bored conference-goer$^{\downarrow}$ moves$^{\uparrow}$'. The $\uparrow$ represents that in order to make a valid inference, the constituent could be replaced by the same or a more general constituent. Conversely, the $\downarrow$ represents that the constituent could be replaced by the same or a less general constituent. Additionally, $=$ represents that the constituent could be replaced by the same constituent. These are what we call *polarities*. Note that the polarity of a given constituent isn't intrinsic to the constituent and differs by the sentence. For example, in the sentence 'some conference-goers$^{\uparrow}$ fidget$^{\uparrow}$', the polarity of the word 'conference-goer' is no longer $\downarrow$ as it was in the example above but rather $\uparrow$.

With the goal of algorithmically implementing monotonicity reasoning, previous work has turned to Combinatory Categorial Grammer (CCG). In CCG, constituents are associated with a syntactic category, such as S (sentence), N (noun), and NP (noun phrase). However, syntactic categories can also be complex, such as S/NP, which denotes a function accepting a NP and returning a S. Combinatory rules, such as > (application), T (type-raising), B (composition), and S (substitution), allow the constituents to be combined. [6] In this way, CCG can be used to parse language. CCG is an ideal language parser for monotonicity reasoning because CCG relies on combinator functions and monotonicity reasoning is a property of functions [4]. Thus, given a sentence, knowledge of the polarities of its constituents, and knowledge of how monotonicity reasoning works with the combinator rules used in CCG, natural language inferences could be made algorithmically.

To understand how monotonicity reasoning works with combinators, we need to first introduce several definitions, taken from [5] and [3].

**Definition 1.1.** *A preorder $\mathbb{P} = (\mathbb{P}, \leq)$ is a set $\mathbb{P}$ with a relation $\leq$ on $\mathbb{P}$ which is reflexive and transitive.*

**Definition 1.2.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be any preorders, and $f : \mathbb{P} \to \mathbb{Q}$ be a function.*

1. *We say that $f$ is monotone if $f(x_1) \leq f(x_2)$ for all $x_1 \leq x_2 \in \mathbb{P}$. We write $f : \mathbb{P} \xrightarrow{+} \mathbb{Q}$ to denote that $f$ is monotone.*
2. *We say that $f$ is antitone if $f(x_2) \leq f(x_1)$ for all $x_1 \leq x_2 \in \mathbb{P}$. We write $f : \mathbb{P} \xrightarrow{-} \mathbb{Q}$ to denote that $f$ is antitone.*
3. *We write $f : \mathbb{P} \xrightarrow{\cdot} \mathbb{Q}$ to denote that $f$ is any function.*

**Definition 1.3.** *The set $Mar$ of markings is given by $Mar = \{+, -, \cdot\}$. We use letters like $m, n$, and $l$ as marking variables ranging over the set $Mar$.*

**Definition 1.4.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be any preorders, and $f : \mathbb{P} \to \mathbb{Q}$ be a function.*

(1) *We write* $x^{\uparrow} : \mathbb{P}$ *to denote that* $x_1 \leq x_2 \in \mathbb{P}$. *Similarly, we write* $f^{\uparrow} : \mathbb{P} \to \mathbb{Q}$ *to denote that* $f_1 \leq f_2$ *pointwise. That is,* $f_1(x) \leq f_2(x) \in \mathbb{Q}$ *for all* $x \in \mathbb{P}$.
(2) *We write* $x^{\downarrow} : \mathbb{P}$ *to denote that* $x_2 \leq x_1 \in \mathbb{P}$. *Similarly, we write* $f^{\downarrow} : \mathbb{P} \to \mathbb{Q}$ *to denote that* $f_2 \leq f_1$ *pointwise. That is,* $f_2(x) \leq f_1(x) \in \mathbb{Q}$ *for all* $x \in \mathbb{P}$.
(3) *We write* $x^{=} : \mathbb{P}$ *to denote that* $x_1 = x_2 \in \mathbb{P}$. *Similarly, we write* $f^{=} : \mathbb{P} \to \mathbb{Q}$ *to denote that* $f_1 = f_2$ *pointwise. That is,* $f_1(x) = f_2(x) \in \mathbb{Q}$ *for all* $x \in \mathbb{P}$.

**Definition 1.5.** *The set* Pol *of* polarities *is given by* Pol $= \{\uparrow, \downarrow, =\}$. *We use* $d$ *as a polarity variable ranging over the set Pol.*

**Definition 1.6.** *A* polarity expression *is either a word* $w = m_1\ m_2\ \cdots\ m_k$ *where the* $m_i$*'s are markings or marking variables and* $k \geq 1$ *or* $w = m_1\ m_2\ \cdots\ m_k\ d$ *where the* $m_i$*'s are markings or marking variables,* $d$ *is a polarity or polarity variable, and* $k \geq 0$. *For example,* $mn$ *and* $+-\downarrow$ *are polarity expressions.*

**Definition 1.7.** *We "multiply" markings and polarities in polarity expressions (in any order) according to the following charts:*

| $m \backslash n$ | $+$ | $-$ | $\cdot$ |
|---|---|---|---|
| $+$ | $+$ | $-$ | $\cdot$ |
| $-$ | $-$ | $+$ | $\cdot$ |
| $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |

| $m \backslash d$ | $\uparrow$ | $\downarrow$ | $=$ |
|---|---|---|---|
| $+$ | $\uparrow$ | $\downarrow$ | $=$ |
| $-$ | $\downarrow$ | $\uparrow$ | $=$ |
| $\cdot$ | $=$ | $=$ | $=$ |

**Definition 1.8.** *A* polarized term *is a term* $t$ *together with a polarity expression, which we usually write as a superscript. For example,* $x^{\uparrow} : \mathbb{P}$ *and* $x^{mnd} : \mathbb{P}$ *are both polarized terms, as is* $f^{md} : \mathbb{P} \xrightarrow{n} \mathbb{Q}$.

**Definition 1.9.** *A* proper typed combinator *is a term* $\lambda x_1 : \sigma_1.\lambda x_2 : \sigma_2.\cdots\lambda x_n : \sigma_n.t$ *where* $t$ *is a term built from* $x_1, x_2, \ldots, x_n$ *using application only.*

**Definition 1.10.** *A* polarized combinator *is a proper typed combinator with all of its terms polarized.*

With these definitions in mind, we can now look at the application combinator rule, which we write as

$$\frac{f : \mathbb{P} \to \mathbb{Q} \quad x : \mathbb{P}}{f(x) : \mathbb{Q}}$$

to denote the function $f$ accepting the input $x$ and returning $f(x)$ as its output, and apply monotonicity reasoning to it as done in [3]:

**Proposition 1.1.** *Let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be any preorders. Then,*

(1) *If* $f_1 = f_2 \in \mathbb{P} \xrightarrow{\cdot} \mathbb{Q}$ *and* $x_1 = x_2 \in \mathbb{P}$, *then* $f_1(x_1) = f_2(x_2) \in \mathbb{Q}$.
(2) *If* $f_1 = f_2 \in \mathbb{P} \xrightarrow{+} \mathbb{Q}$ *and* $x_1 = x_2 \in \mathbb{P}$, *then* $f_1(x_1) = f_2(x_2) \in \mathbb{Q}$.
(3) *If* $f_1 = f_2 \in \mathbb{P} \xrightarrow{-} \mathbb{Q}$ *and* $x_1 = x_2 \in \mathbb{P}$, *then* $f_1(x_1) = f_2(x_2) \in \mathbb{Q}$.
(4) *If* $f_1 \leq f_2 \in \mathbb{P} \xrightarrow{\cdot} \mathbb{Q}$ *and* $x_1 = x_2 \in \mathbb{P}$, *then* $f_1(x_1) \leq f_2(x_2) \in \mathbb{Q}$.
(5) *If* $f_1 \leq f_2 \in \mathbb{P} \xrightarrow{+} \mathbb{Q}$ *and* $x_1 \leq x_2 \in \mathbb{P}$, *then* $f_1(x_1) \leq f_2(x_2) \in \mathbb{Q}$.
(6) *If* $f_1 \leq f_2 \in \mathbb{P} \xrightarrow{-} \mathbb{Q}$ *and* $x_2 \leq x_1 \in \mathbb{P}$, *then* $f_1(x_1) \leq f_2(x_2) \in \mathbb{Q}$.
(7) *If* $f_2 \leq f_1 \in \mathbb{P} \xrightarrow{\cdot} \mathbb{Q}$ *and* $x_1 = x_2 \in \mathbb{P}$, *then* $f_2(x_2) \leq f_1(x_1) \in \mathbb{Q}$.
(8) *If* $f_2 \leq f_1 \in \mathbb{P} \xrightarrow{+} \mathbb{Q}$ *and* $x_2 \leq x_1 \in \mathbb{P}$, *then* $f_2(x_2) \leq f_1(x_1) \in \mathbb{Q}$.
(9) *If* $f_2 \leq f_1 \in \mathbb{P} \xrightarrow{-} \mathbb{Q}$ *and* $x_1 \leq x_2 \in \mathbb{P}$, *then* $f_2(x_2) \leq f_1(x_1) \in \mathbb{Q}$.

The nine cases in this proposition, which we will later prove in Proposition 2.1, was summarized in [2] by polarizing the application combinator as follows:

**Rule 1** (Polarized Application)**.**

$$\frac{f^{d} : \mathbb{P} \xrightarrow{m} \mathbb{Q} \quad x^{md} : \mathbb{P}}{f(x)^{d} : \mathbb{Q}}$$

A similar process could be used to polarize other combinators; monotonicity reasoning could be worked through in cases and then summarized into one rule. However, this is a tedious process, especially as the number of cases and the complexity of each cases increases with more complex combinators. Thus, this paper provides a proof system that builds inductively on the polarized application rule and the following polarized withdrawal rule to simplify the process of polarizing typed combinators.

**Rule 2** (Polarized Withdrawal)**.**

$$\dfrac{\dfrac{\dfrac{[x^{md} : \sigma_1]}{\vdots \qquad \vdots}}{y(x)^{d} : \sigma_2}}{\lambda x.y(x)^{d} : \sigma_1 \xrightarrow{m} \sigma_2}$$

## 2. The Proof System

In order to build inductively on Rule 1, our proof system accepts a proper typed combinator written as an application tree. For example, the S Combinator

$$\dfrac{x : a \to b \to c \quad y : a \to b}{(\mathsf{s}xy) : a \to c}\ \mathsf{S}$$

can be written as the following application tree:

$$\dfrac{\dfrac{x : a \to b \to c \quad z : a}{(xz) : b \to c} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{xz(yz) : c}$$

This is possible because recalling from Definition 1.9, proper typed combinators are built using the application combinator.

From this application tree, Rule 1 can be used to build an application proof tree.

**Definition 2.1.** *An* application proof tree *is a finite full binary tree whose nodes are labeled with polarized terms $t$ and*

1. *Every internal node is an instance of the polarized application rule*
2. *Every leaf instance of a given variable has an identical polarity expression.*

*We write*

$$t_1, t_2, \dots, t_n \vdash u$$

*if there is some proof tree all of whose leaves are among $t_1$, …, $t_n$, and whose root is $u$.*

Then, Rule 2 can be used to build a complete proof tree.

**Definition 2.2.** *A* complete proof tree *is a finite binary tree whose nodes are labeled with polarized terms $t$ and*

1. *The tree contains a full binary application subtree adhering to Definition 2.1.*
2. *Every leaf node of the full binary subtree is withdrawn*
3. *Every node below the full binary subtree is an instance of the polarized withdrawal rule.*

Thus, to consider the soundness of the proof system, we must first consider the soundness of both rules. Doing so will require additional definitions.

**Definition 2.3.** *A* structure $\mathbb{D}$ *is given by: an assignment $\sigma \mapsto \mathbb{D}_\sigma$ taking types to preorders. We write $\mathbb{D}_\sigma = (D_\sigma, \leq_\sigma)$ when we need to exhibit parts of the preorder structures. For the base types $\mathbb{D}_b$ is arbitrary. But for complex types, we have some requirements:*

$$\begin{array}{lcl} \mathbb{D}_{\sigma_1 \xrightarrow{\cdot} \sigma_2} & = & (D_{\sigma_2})^{D_{\sigma_1}} \text{ i.e., all functions from the set } D_{\sigma_1} \text{ to the set } D_{\sigma_2} \\ \mathbb{D}_{\sigma_1 \xrightarrow{+} \sigma_2} & = & \{f \in \mathbb{D}_{\sigma_1 \xrightarrow{\cdot} \sigma_2} : f \text{ is monotone}\} \\ \mathbb{D}_{\sigma_1 \xrightarrow{-} \sigma_2} & = & \{f \in \mathbb{D}_{\sigma_1 \xrightarrow{\cdot} \sigma_2} : f \text{ is antitone}\} \end{array}$$

*In all of these cases, the ordering on the function set is the pointwise order.*

**Definition 2.4.** *Let $t : \sigma$ be a typed term (without marking variables). Let $\mathbb{D}$ be a structure, and let $\llbracket\ \rrbracket_1$ and $\llbracket\ \rrbracket_2$ be two interpretations in $\mathbb{D}$. Then,*

$$\begin{array}{lll} \llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models t^{\uparrow} : \sigma & \textit{means} & \llbracket t \rrbracket_1 \leq \llbracket t \rrbracket_2 \textit{ in } \mathbb{D}_\sigma \\ \llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models t^{\downarrow} : \sigma & \textit{means} & \llbracket t \rrbracket_2 \leq \llbracket t \rrbracket_1 \textit{ in } \mathbb{D}_\sigma \\ \llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models t^{=} : \sigma & \textit{means} & \llbracket t \rrbracket_1 = \llbracket t \rrbracket_2 \textit{ in } \mathbb{D}_\sigma \end{array}$$

**Definition 2.5.** *A proof system is* sound *if for all $\varphi$ such that $\Gamma \vdash \varphi$, we have $\Gamma \models \varphi$. That is, if we can prove $\varphi$ from $\Gamma$ using the rules of the proof system, there are no possible models such that $\Gamma$ is true but $\varphi$ is false.*

**Proposition 2.1** (Soundness of Polarized Application)**.**

$$f^{d} : \sigma_1 \stackrel{m}{\to} \sigma_2, x^{md} : \sigma_1 \models (f(x))^{d} : \sigma_2 \tag{1}$$

*Proof.* Let $\mathbb{D}$ be a structure and take $\llbracket\ \rrbracket_1$ and $\llbracket\ \rrbracket_2$ to be two interpretations in this structure $\mathbb{D}$. Then,

Case 1: Let $d$ be $=$ and $m$ be $\cdot$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{=} : \sigma_1 \stackrel{\cdot}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{=} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_1 = \llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$. Since, by Definition 2.3, the ordering on $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$ is the pointwise order, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) = \llbracket f \rrbracket_2(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$. Furtheremore, since $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$, we know that $\llbracket f \rrbracket_2(\llbracket x \rrbracket_1) = \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$. Thus, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) = \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$, implying by Definition 2.3 that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{=} : \sigma_2$ as desired.

Case 2: Let $d$ be $=$ and $m$ be $+$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{=} : \sigma_1 \stackrel{+}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{=} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_1 = \llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{+}{\to} \sigma_2}$. This gives us the same rule as in Case 1 with the exception that here $\mathbb{D}_{\sigma_1 \stackrel{+}{\to} \sigma_2}$ instead of $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$ as in Case 1. However, since we only this fact to state that the order on $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$ is the pointwise order and the same is true for $\mathbb{D}_{\sigma_1 \stackrel{+}{\to} \sigma_2}$, we can again conclude that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{=} : \sigma_2$ as desired.

Case 3: Let $d$ be $=$ and $m$ be $-$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{=} : \sigma_1 \stackrel{-}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{=} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_1 = \llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{-}{\to} \sigma_2}$. This gives us the same rule as in Case 1 with the exception that here $\mathbb{D}_{\sigma_1 \stackrel{-}{\to} \sigma_2}$ instead of $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$ as in Case 1. However, since we only this fact to state that the order on $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$ is the pointwise order and the same is true for $\mathbb{D}_{\sigma_1 \stackrel{-}{\to} \sigma_2}$, we can again conclude that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{=} : \sigma_2$ as desired.

Case 4: Let $d$ be $\uparrow$ and $m$ be $\cdot$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{\uparrow} : \sigma_1 \stackrel{\cdot}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{=} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_1 \leq \llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$. Since, by Definition 2.3, the ordering on $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$ is the pointwise order, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$. Furtheremore since $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$, we know that $\llbracket f \rrbracket_2(\llbracket x \rrbracket_1) = \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$. Thus, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$, which implies by Definition 2.3 that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{\uparrow} : \sigma_2$ as desired.

Case 5: Let $d$ be $\uparrow$ and $m$ be $+$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{\uparrow} : \sigma_1 \stackrel{+}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{\uparrow} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_1 \leq \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_1 \leq \llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{+}{\to} \sigma_2}$. Since, by Definition 2.3, the ordering on $\mathbb{D}_{\sigma_1 \stackrel{+}{\to} \sigma_2}$ is the pointwise order, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$. Furtheremore, by Definition 2.3, $\llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{+}{\to} \sigma_2}$ is monotone, implying that since $\llbracket x \rrbracket_1 \leq \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$, we know that $\llbracket f \rrbracket_2(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$. Thus, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$, which implies by Definition 2.3 that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{\uparrow} : \sigma_2$ as desired.

Case 6: Let $d$ be $\uparrow$ and $m$ be $-$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{\uparrow} : \sigma_1 \stackrel{-}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{\downarrow} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_2 \leq \llbracket x \rrbracket_1$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_1 \leq \llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{-}{\to} \sigma_2}$. Since, by Definition 2.3, the ordering on $\mathbb{D}_{\sigma_1 \stackrel{-}{\to} \sigma_2}$ is the pointwise order, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$. Furtheremore, by Definition 2.3, $\llbracket f \rrbracket_2$ in $\mathbb{D}_{\sigma_1 \stackrel{-}{\to} \sigma_2}$ is antitone, implying that since $\llbracket x \rrbracket_2 \leq \llbracket x \rrbracket_1$ in $\mathbb{D}_{\sigma_1}$, we know that $\llbracket f \rrbracket_2(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$. Thus, $\llbracket f \rrbracket_1(\llbracket x \rrbracket_1) \leq \llbracket f \rrbracket_2(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$, which implies by Definition 2.3 that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{\uparrow} : \sigma_2$ as desired.

Case 7: Let $d$ be $\downarrow$ and $m$ be $\cdot$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{\downarrow} : \sigma_1 \stackrel{\cdot}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{=} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_2 \leq \llbracket f \rrbracket_1$ in $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$. Since, by Definition 2.3, the ordering on $\mathbb{D}_{\sigma_1 \stackrel{\cdot}{\to} \sigma_2}$ is the pointwise order, $\llbracket f \rrbracket_2(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$. Furtheremore since $\llbracket x \rrbracket_1 = \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$, we know that $\llbracket f \rrbracket_1(\llbracket x \rrbracket_2) = \llbracket f \rrbracket_1(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$. Thus, $\llbracket f \rrbracket_2(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$, which implies by Definition 2.3 that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{\downarrow} : \sigma_2$ as desired.

Case 8: Let $d$ be $\downarrow$ and $m$ be $+$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{\downarrow} : \sigma_1 \overset{+}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{\downarrow} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_2 \leq \llbracket x \rrbracket_1$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_2 \leq \llbracket f \rrbracket_1$ in $\mathbb{D}_{\sigma_1 \overset{+}{\to} \sigma_2}$. Since, by Definition 2.3, the ordering on $\mathbb{D}_{\sigma_1 \overset{+}{\to} \sigma_2}$ is the pointwise order, $\llbracket f \rrbracket_2(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$. Furthermore, by Definition 2.3, $\llbracket f \rrbracket_1$ in $\mathbb{D}_{\sigma_1 \overset{+}{\to} \sigma_2}$ is monotone, implying that since $\llbracket x \rrbracket_2 \leq \llbracket x \rrbracket_1$ in $\mathbb{D}_{\sigma_1}$, we know that $\llbracket f \rrbracket_1(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$. Thus, $\llbracket f \rrbracket_2(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$, which implies by Definition 2.3 that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{\downarrow} : \sigma_2$ as desired.

Case 9: Let $d$ be $\downarrow$ and $m$ be $-$, giving us $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models f^{\downarrow} : \sigma_1 \overset{-}{\to} \sigma_2$ and $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models x^{\uparrow} : \sigma_1$. By Definition 2.4, this means that $\llbracket x \rrbracket_1 \leq \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$ and that $\llbracket f \rrbracket_2 \leq \llbracket f \rrbracket_1$ in $\mathbb{D}_{\sigma_1 \overset{-}{\to} \sigma_2}$. Since, by Definition 2.3, the ordering on $\mathbb{D}_{\sigma_1 \overset{-}{\to} \sigma_2}$ is the pointwise order, $\llbracket f \rrbracket_2(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_2)$ in $\mathbb{D}_{\sigma_2}$. Furthermore, by Definition 2.3, $\llbracket f \rrbracket_1$ in $\mathbb{D}_{\sigma_1 \overset{+}{\to} \sigma_2}$ is antitone, implying that since $\llbracket x \rrbracket_1 \leq \llbracket x \rrbracket_2$ in $\mathbb{D}_{\sigma_1}$, we know that $\llbracket f \rrbracket_1(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$. Thus, $\llbracket f \rrbracket_2(\llbracket x \rrbracket_2) \leq \llbracket f \rrbracket_1(\llbracket x \rrbracket_1)$ in $\mathbb{D}_{\sigma_2}$, which implies by Definition 2.3 that $\llbracket\ \rrbracket_1, \llbracket\ \rrbracket_2 \models (f(x))^{\downarrow} : \sigma_2$ as desired.

This concludes all possible cases of the polarized application rule. □

With this, we can prove the soundness of the application proof tree.

**Lemma 2.1** (Soundness of the Application Proof Tree)**.** *Base case: When we have an application proof tree comprised of just a polarized root, it is trivially sound.*
*Inductive hypotheses: Assume that there is a sound application proof tree comprised of k polarized nodes. Then, if we add two polarized child nodes that respect the criteria of Definition 2.1, we know by Proposition 2.1 that the new polarized application sub-tree created by the added nodes is sound. Thus, since we also know by our assumption that the rest of the application proof tree is sound as well, we know that we have a sound application proof tree.*

However, in this paper we will only conjecture that the polarized withdrawal rule is sound, and leave its proof for future work.

**Conjecture 2.1** (Soundness of Polarized Withdrawal)**.**

Thus, assuming the truth of this conjecture, we can conditionally prove the soundness of the proof system.

**Conjecture 2.2** (Soundness of the Complete Proof Tree)**.** *Base case: When we have an application proof tree, we know by Lemma 2.1 that it is sound.*
*Inductive hypotheses: Assume that there is a sound application proof tree. Then, if we add a parent node such that we respect the criteria of Definition 2.2, we know by our assumption that Proposition 2.1 is true, that the new polarized withdrawal sub-tree created by the added node is sound. Thus, since we also know by our assumption that the rest of the proof tree is sound as well, we know that we have a sound proof tree.*
*By induction, this implies that when we withdraw all leaf nodes of the application proof tree, giving us a complete proof tree, it will be sound as well.*

## 3. The Algorithm

The following is an algorithm for implementing the proof system for a subset of proper typed combinators. Subsections 3.1 and 3.2 illustrate examples in which the algorithm successfully implements the proof system. However, Subsection 3.3 illustrates an example where the algorithm fails. Ultimately, we are left with an open question about when this algorithm does and doesn't work and whetehr we can improve it to work for all proper typed combinators.

**Algorithm 1.** (1) *raise the root of the tree to the d polarity variable*
(2) *let $h$ = the height of the tree and let $i = 0$*
(3) ***while** $i < h$:*
(4) ***for** all subtrees on levels $i$ and $i+1$, working from right to left:*
(5) *raise the left child node to the same polarity expression that the parent node is raised to*
(6) ***if** the right child node is a leaf and is identical to another leaf that has already been raised to a polarity expression:*
(7) *raise it to the same polarity expression as the identical leaf is raised to*
(8) ***else:***

(9) *raise the right child node to a new marking variable multiplied by the same polarity expression that the parent node is raised to*
(10) ***end if***
(11) *mark the main application arrow of the left child node with all marking variables present in the polarity expression that the right child node is raised to but not present in the polarity expression that the parent node is raised to*
(12) *mark any other arrows in the left child node that have already been marked below with the same marking variables as marked below*
(13) ***end for***
(14) *let* $i = i + 1$
(15) ***end while***
(16) *let* $k =$ *the number of unique leaf terms*
(17) ***while*** $k \neq 0$***:***
(18) *bracket all leaf instances of the* $k$*th term*
(19) *under the previous root, create a new root with a function from the bracketed leaf term to the previous root term, maintaining the marking variables on any arrows*
(20) *mark the main arrow of the new root term with the marking variables of the polarity expression that the bracketed leaf term is raised to or with a + if there are no marking variables in the polarity expression*
(21) *raise the new root term to the same polarity expression as the previous root term*
(22) *let* $k = k - 1$
(23) ***end while***
(24) *pull out the polarized nodes necessary for writing the combinator rule*

### 3.1. Application of the Algorithm.

We were able to use this algorithm to re-obtain the following polarized combinators which were previously obtained in [5] by tediously working through each case. Note that these polarized combinators have been assumed to cover each correct concrete case.

$$\frac{f^{d} : a \xrightarrow{m} b \quad x^{md} : a}{(fx)^{d} : b} >$$

$$\frac{f^{nd} : a \xrightarrow{m} b \quad g^{d} : b \xrightarrow{n} c}{(\mathsf{B}gf)^{d} : a \xrightarrow{mn} c} \mathsf{B}$$

$$\frac{x^{d} : a \xrightarrow{mn} b \xrightarrow{n} c \quad y^{nd} : a \xrightarrow{m} b}{(\mathsf{S}xy)^{d} : a \xrightarrow{mn} c} \mathsf{S}$$

$$\frac{p^{md} : a}{(\mathsf{T}p)^{d} : a \xrightarrow{m} b \xrightarrow{+} b} \mathsf{T}$$

### 3.2. Example with the S Combinator.

To demonstrate how this algorithm works, we'll begin with the S combinator as an application tree.

$$\frac{\dfrac{x : a \to b \to c \quad z : a}{(xz) : b \to c} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{xz(yz) : c}$$

Then, as per Step 1, we'll raise the root of the tree to the $d$ polarity variable.

$$\frac{\dfrac{x : a \to b \to c \quad z : a}{(xz) : b \to c} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{xz(yz)^{d} : c}$$

Working through Steps 2–4, we have $i = 0 < h = 2$ and only one subtree on levels 0 and 1, so following Step 5, we'll raise the left child node of that subtree to the $d$ polarity variable just like the parent node.

$$\dfrac{\dfrac{x : a \to b \to c \quad z : a}{(xz)^{d} : b \to c} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{xz(yz)^{d} : c}$$

Since the right child node of that subtree is not a leaf, we'll follow Step 9 and raise it to a new marking variable $n$ multiplied by the $d$ polarity variable that the parent node is raised to.

$$\dfrac{\dfrac{x : a \to b \to c \quad z : a}{(xz)^{d} : b \to c} \quad \dfrac{y : a \to b \quad z : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

As per Step 11, we'll mark the application arrow of the left child node of that subtree with the new $n$ marking variable.

$$\dfrac{\dfrac{x : a \to b \to c \quad z : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y : a \to b \quad z : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

As per Steps 14 and 3, we now have $i = 1 < h = 2$, so following Step 4, we'll focus on the right subtree on levels 1 and 2. Following Step 5, we'll raise the left child node of that subtree to the $nd$ polarity expression that the parent node is raised to.

$$\dfrac{\dfrac{x : a \to b \to c \quad z : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \to b \quad z : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

Since the right child node is not identical to another leaf that has already been marked and polarized, we'll follow Step 9 and raise it to a new marking variable $m$ multiplied by the $nd$ polarity expression that the parent node is raised to.

$$\dfrac{\dfrac{x : a \to b \to c \quad z : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \to b \quad z^{mnd} : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

As per Step 11, we'll mark the application arrow of the left child node of that subtree with the new $m$ marking variable.

$$\dfrac{\dfrac{x : a \to b \to c \quad z : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad z^{mnd} : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

Following Step 4, we'll now focus on the left subtree on levels 1 and 2. Following Step 6, we'll raise the left child node of that subtree to the $d$ polarity variable that the parent node is raised to.

$$\dfrac{\dfrac{x^{d} : a \to b \to c \quad z : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad z^{mnd} : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

Since the right child node is a leaf and is identical to a leaf of the right subtree that was previously raised to the $mnd$ polarity expression, we'll follow Step 8 and raise it to the same polarity expression.

$$\dfrac{\dfrac{x^{d} : a \to b \to c \quad z^{mnd} : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad z^{mnd} : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

As per Step 11, we'll mark the main application arrow of the left child node of that subtree with the $mn$ marking expression.

$$\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \to c \quad z^{mnd} : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad z^{mnd} : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

Then, following Step 12, we'll mark the other arrow in the left child node as it was marked previously.

$$\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad z^{mnd} : a}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad z^{mnd} : a}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

Following Steps 14 and 3, we now have $i = 2 \not< h = 2$, so we can now move on to Steps 16 and 17 where $k = 3$. Thus, as per Step 18, we'll bracket all instances of the $z$ term.

$$\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}$$

As per Step 19, we'll create a new root under the previous root with a function from the bracketed leaf term to the previous root term.

$$\dfrac{\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz)) : a \to c}$$

As per Step 20, we'll mark the main arrow of the new root term with the marking variables $mn$ of the polarity expression that the $z$ term is raised to.

$$\dfrac{\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz)) : a \overset{mn}{\to} c}$$

As per Step 21, we'll raise the new root term to the same $d$ polarity variable as the previous root term.

$$\dfrac{\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{y^{nd} : a \overset{m}{\to} b \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}$$

Following Steps 22 and 17, we now have $k = 2$, so as per Step 18, we'll bracket the $y$ term.

$$\dfrac{\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}$$

As per Step 19, we'll create a new root under the previous root with a function from the bracketed leaf term to the previous root term.

$$\dfrac{\dfrac{\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \qquad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}}{(\lambda y.\lambda z.xz(yz)) : (a \overset{m}{\to} b) \to a \overset{mn}{\to} c}$$

As per Step 20, we'll mark the main arrow of the new root term with the marking variable $n$ of the polarity expression that the $y$ term is raised to.

$$\dfrac{\dfrac{\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \qquad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}}{(\lambda y.\lambda z.xz(yz)) : (a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c}$$

As per Step 21, we'll raise the new root term to the same $d$ polarity variable as the previous root term.

$$\dfrac{\dfrac{\dfrac{\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \qquad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}}{(\lambda y.\lambda z.xz(yz))^{d} : (a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c}$$

Following Steps 22 and 17, we now have $k = 1$, so as per Step 18, we'll bracket the $x$ term.

$$\dfrac{\dfrac{\dfrac{[x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c] \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \qquad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}$$

As per Step 19, we'll create a new root under the previous root with a function from the bracketed leaf term to the previous root term.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c] \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \qquad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}}{(\lambda y.\lambda z.xz(yz))^{d} : (a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c}}{(\lambda x.\lambda y.\lambda z.xz(yz)) : (a \overset{mn}{\to} b \overset{n}{\to} c) \to ((a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c)}$$

As per Step 20, we'll mark the main arrow of the new root term with a + since there are no marking variables in the polarity expression that the $y$ term is raised to.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c] \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \qquad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}}{(\lambda y.\lambda z.xz(yz))^{d} : (a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c}}{(\lambda x.\lambda y.\lambda z.xz(yz)) : (a \overset{mn}{\to} b \overset{n}{\to} c) \overset{+}{\to} ((a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c)}$$

Finally, as per Step 21, we'll, we'll raise the new root term to the same $d$ polarity variable as the previous root term.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c] \quad [z^{mnd} : a]}{(xz)^{d} : b \overset{n}{\to} c} \quad \dfrac{[y^{nd} : a \overset{m}{\to} b] \quad [z^{mnd} : a]}{(yz)^{nd} : b}}{xz(yz)^{d} : c}}{(\lambda z.xz(yz))^{d} : a \overset{mn}{\to} c}}{(\lambda y.\lambda z.xz(yz))^{d} : (a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c}}{(\lambda x.\lambda y.\lambda z.xz(yz))^{d} : (a \overset{mn}{\to} b \overset{n}{\to} c) \overset{+}{\to} ((a \overset{m}{\to} b) \overset{n}{\to} a \overset{mn}{\to} c)}$$

Finally, Step 24 gives us the following rule:

$$\dfrac{x^{d} : a \overset{mn}{\to} b \overset{n}{\to} c \quad y^{nd} : a \overset{m}{\to} b}{(\mathsf{s}xy)^{d} : a \overset{mn}{\to} c}\ \mathsf{s}$$

3.3. **Counterexample.** However, to illustrate why this algorithm sometimes fails to implement the proof system, we'll consider the combinator which can be written with the following application tree:

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz)) : c \to d} \quad \dfrac{w : a \to c \quad z : a}{(wz) : c}}{((xz)(yz))(wz) : d}$$

Then, as per Step 1, we'll raise the root of the tree to the $d$ polarity variable.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz)) : c \to d} \quad \dfrac{w : a \to c \quad z : a}{(wz) : c}}{((xz)(yz))(wz)^{d} : d}$$

Working through Steps 2–4, we have $i = 0 < h = 3$ and one subtree on levels 0 and 1, so following Step 5, we'll raise the left child node of that subtree to the $d$ polarity variable just like the parent node.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz))^{d} : c \to d} \quad \dfrac{w : a \to c \quad z : a}{(wz) : c}}{((xz)(yz))(wz)^{d} : d}$$

Since the right child node of that subtree is not a leaf, we'll follow Step 9 and raise it to a new marking variable $n$ multiplied by the $d$ polarity variable that the parent node is raised to.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz))^{d} : c \to d} \quad \dfrac{w : a \to c \quad z : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

As per Step 11, we'll mark the application arrow of the left child node of that subtree with the new $n$ marking variable.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w : a \to c \quad z : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

As per Steps 14 and 3, we now have $i = 1 < h = 3$, so following Step 4, we'll focus on the right subtree on levels 1 and 2. Following Step 5, we'll raise the left child node of that subtree to the $nd$ polarity expression that the parent node is raised to.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w^{nd} : a \to c \quad z : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

Since the right child node is not identical to another leaf that has already been marked and polarized, we'll follow Step 9 and raise it to a new marking variable $m$ multiplied by the $nd$ polarity expression that the parent node is raised to.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w^{nd} : a \to c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

As per Step 11, we'll mark the application arrow of the left child node of that subtree with the new $m$ marking variable.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz) : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w^{nd} : a \overset{m}{\to} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

Following Step 4, we'll now focus on the left subtree on levels 1 and 2. Following Step 6, we'll raise the left child node of that subtree to the $d$ polarity variable that the parent node is raised to.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz)^{d} : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz) : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w^{nd} : a \overset{m}{\to} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

Since the right child node of that subtree is not a leaf, we'll follow Step 9 and raise it to a new marking variable $l$ multiplied by the $d$ polarity variable that the parent node is raised to.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz)^{d} : b \to c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz)^{ld} : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w^{nd} : a \overset{m}{\to} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

As per Step 11, we'll mark the main application arrow of the left child node of that subtree with the $l$ marking expression.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz)^{d} : b \overset{l}{\to} c \to d} \quad \dfrac{y : a \to b \quad z : a}{(yz)^{ld} : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w^{nd} : a \overset{m}{\to} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

Then, following Step 12, we'll mark the other arrow in the left child node as it was marked previously.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz)^{d} : b \overset{l}{\to} c \overset{n}{\to} d} \quad \dfrac{y : a \to b \quad z : a}{(yz)^{ld} : b}}{(xz(yz))^{d} : c \overset{n}{\to} d} \quad \dfrac{w^{nd} : a \overset{m}{\to} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

Following Steps 14 and 3, we now have $i = 2 < h = 3$, so following Step 4, we'll focus on the right subtree on levels 2 and 3. Following Step 5, we'll raise the left child node of that subtree to the $ld$ polarity expression that the parent node is raised to.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz)^{d} : b \xrightarrow{l} c \xrightarrow{n} d} \quad \dfrac{y^{ld} : a \to b \quad z : a}{(yz)^{ld} : b}}{(xz(yz))^{d} : c \xrightarrow{n} d} \quad \dfrac{w^{nd} : a \xrightarrow{m} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

Since the right child node is a leaf and is identical to a leaf of the right subtree that was previously raised to the $mnd$ polarity expression, we'll follow Step 8 and raise it to the same polarity expression.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz)^{d} : b \xrightarrow{l} c \xrightarrow{n} d} \quad \dfrac{y^{ld} : a \to b \quad z^{mnd} : a}{(yz)^{ld} : b}}{(xz(yz))^{d} : c \xrightarrow{n} d} \quad \dfrac{w^{nd} : a \xrightarrow{m} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

As per Step 11, we'll mark the main application arrow of the left child node of that subtree with the $mn$ marking expression.

$$\dfrac{\dfrac{\dfrac{x : a \to b \to c \to d \quad z : a}{(xz)^{d} : b \xrightarrow{l} c \xrightarrow{n} d} \quad \dfrac{y^{ld} : a \xrightarrow{mn} b \quad z^{mnd} : a}{(yz)^{ld} : b}}{(xz(yz))^{d} : c \xrightarrow{n} d} \quad \dfrac{w^{nd} : a \xrightarrow{m} c \quad z^{mnd} : a}{(wz)^{nd} : c}}{((xz)(yz))(wz)^{d} : d}$$

However, notice that this gives us the following subtree, which is not an instance of the polarized application rule.

$$\dfrac{y^{ld} : a \xrightarrow{mn} b \quad z^{mnd} : a}{(yz)^{ld} : b}$$

Thus, the algorithm fails to implement the proof system for this combinator.

## 4. Conclusion

In this paper, we've presented a proof system for polarizing proper typed combinators which builds inductively on two rules. While we proved the soundness of the polarized application rule, we only conjectured about the soundness of the polarized withdrawal rule, leaving its proof for future work. Consequently, the soundness of the entire proof system hinges on this later work.

Additionally, we provided an algorithm for implementing this proof system that works for a subset of proper typed combinators. This leaves us with an open question about when this algorithm does and doesn't work and whether we can build an algorithm that works for all proper typed combinators.

Furthermore, we have not addressed the completeness of this proof system. We know that the completeness of this system would require additional Mon and Weakening rules described in [5]. However, even with these additional rules it is not known whether this system is complete, leaving this an open question for future research.

Finally, another project left for future work is the computer implementation of the algorithm.

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